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An Absolute Determination of the Acceleration Due to Gravity

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AN ABSOLUTE DETERMINATION OF THE ACCELERATION DUE TO GRAVITY

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CONTENTS

1. Introduction

Recent advances in the precision obtainable in length and time measurements have made it possible to determine the absolute value of the acceleration due to gravity with greater accuracy than has hitherto been possible.

An accurate knowledge of the actual value of the acceleration due to gravity is essential, for example, in connexion with the determination of the absolute unit of electric current by means of the current balance.

It has been usual to refer relative gravity measurements made in this country to the absolute value determined at Potsdam by Kühnen and Furtwängler (1906) over 30 years ago, and although relative determinations can now be carried out with an accuracy approaching one part in a million, it appears that the basic Potsdam value may be in error by something between one and two parts in 100,000. Very few absolute determinations have been made within the last 50 years, but those which have been carried out at various stations show discrepancies of this order when related to the Potsdam value by relative determinations.

Some of the Potsdam determinations were made with reversible pendulums having the knife-edges attached to the pendulum rods, and much difficulty was experienced in measuring the precise distance between the knife-edges. It then appeared, and has subsequently been confirmed, that the length measurements could be carried out with much higher precision if the knife-edges formed a part of the pendulum support and the reversible pendulum itself carried two flat and parallel surfaces which rested in turn on the knife-edges. With modern methods of construction and measurement the distance between two such parallel planes can be measured with very high precision. It was therefore decided to use a reversible pendulum in which the fundamental length measurement consisted simply in determining the distance between the two flat and parallel ends of a bar of simple form, constituting the pendulum rod.

Preliminary experiments were carried out with such a pendulum designed by Professor Kerr Grant of Adelaide University in collaboration with Mr J. E. Sears of The National Physical Laboratory. This pendulum was of steel and consisted of a cylindrical end-gauge 10 in. in length and $\frac{7}{8}$ in. in diameter with flat and parallel ends loaded at each end with a cylindrical "bob", one bob being much lighter than the other and both bobs being provided with adjustments for equalising the periods in the two positions (upright and reversed) of the pendulum. It was found that this pendulum was unsuitable for an absolute determination of gravity owing to the effect of the earth's magnetic field on the period of the steel pendulum. The magnitude of the disturbance was such that it was not considered desirable in the present work to use an invar pendulum which would have been advantageous in some ways had it not been also slightly magnetic.*

A pendulum was then designed having a cylindrical rod 1 m. in length and $\frac{7}{8}$ in. in diameter, to be made in non-magnetic bronze or brass, but on completion of the theoretical investigation described in Appendix III of this paper, it was found that the correction to the period due to the bending of the pendulum rod during oscillation was far too large to be calculable with sufficient accuracy, having regard to the assumptions made in the calculations. Further investigations showed that the correction could be reduced to a more reasonable value by using a hollow cylindrical rod and reducing the weight of the bobs, but it was felt that there might be constructional difficulties in

* That the earth's magnetic field may have an appreciable effect on the period of an invar pendulum has since been confirmed by Bullard (1933).

making a hollow rod with thin walls to the accuracy required for this work. At this stage Mr Sears suggested making the pendulum rod of a non-magnetic light alloy and of I-section. This suggestion was adopted, and the pendulum actually used was designed on this basis. The only length measurement which it is necessary to carry out with high precision is the measurement of the distance between the flat parallel ends of the I-section rod of light alloy. Several mechanical methods are available for determining this length to an accuracy approaching one part in a million and by interferometric methods it is possible to determine it with still greater precision.

With regard to the fundamental time measurement, a precision chronograph (Sears and Tomlinson 1931) is available at the National Physical Laboratory, in which an accuracy of time measurement of the order of 0·0002 second can be attained, and by arranging a suitable electrical contact device which can be brought into action when required for a short time at the beginning and end of any period of time which it is desired to measure, it is possible to determine the mean period of oscillation of the freely swinging pendulum with very high precision in a much shorter time than has hitherto been possible.

2. General description of the apparatus

(*a*) *Pendulum*

As already stated, the reversible pendulum consists of a bar of light alloy (Y-alloy) of I-section and very closely 1 m. in length, to the ends of which are attached rectangular blocks of non-magnetic Delta metal. It is shown diagrammatically in fig. 1.

The external cross-section of the pendulum rod is $1\frac{3}{4} \times 3\frac{3}{4}$ in., and the thickness of the web and flanges is everywhere $\frac{1}{4}$ in. The rod was machined out of a solid block of forged Y-alloy, heat treated for stability, leaving the two ends reinforced as shown in fig. 1, so as to provide a suitable method of attachment for the rectangular blocks, and cut away at A_1 and A_2 so as to permit the insertion of the knife-edge on which the pendulum swings. Each block is held in position by means of four long bolts of non-magnetic bronze, and for convenience in adjusting the blocks so that the period of the pendulum shall be the same in the inverted position as it is in the upright position, the heavier block is made in three pieces, *C*, *D* and *E* in fig. 1. One of these pieces, *C*, is identical with the lighter block, *B*, and consists of a rectangular block of Delta metal $1\frac{3}{4} \times 3\frac{3}{4} \times 1\frac{1}{2}$ in., two opposite surfaces, *aa* and *bb* in fig. 1, being ground and lapped flat and parallel after having been chromium plated. One of these two plated surfaces, *aa* on the block *B*, serves as the plane which rests on the knife-edge, and the other, *bb*, serves as a plane mirror by means of which the amplitude of the pendulum is observed. The block *D* is of the same cross-section but is about $2\frac{3}{8}$ in. in length and is not chromium plated. This piece was reduced in length by repeated trial until the pendulum was found to have very nearly the same period whether it was swinging with the heavy end or the light end above the knife-edge. The third piece, *E*, is also of the same cross-section but its length is $2\frac{1}{4}$ in., making the total length of the heavy bob about $6\frac{1}{8}$ in. This piece is chromium plated on one surface only, *cc*, thus providing a plane mirror for the heavy end of the pendulum by which its amplitude is observed when in the inverted position. Each of the blocks *B* and *C* has bored through it a transverse hole $\frac{9}{32}$ in. diameter through which can be threaded a long steel pin which serves as a convenient method of supporting the pendulum until it is ready to be lowered on to the knife-edge (see § 2 (*d*)).

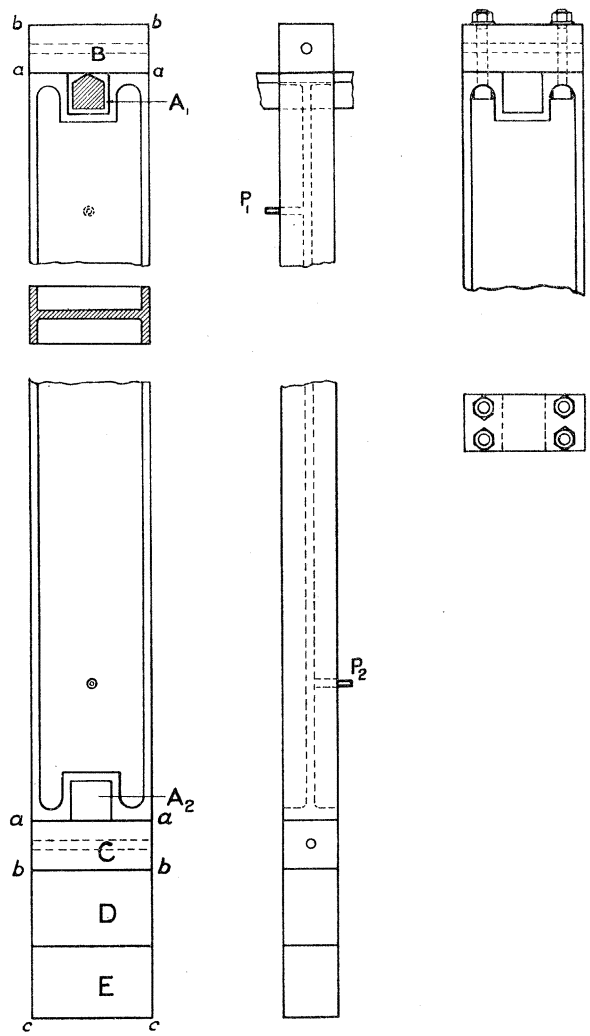

FIG. 1. Reversible pendulum, diagrammatic.

The I-section pendulum rod also carries two projecting pins, P_1 and P_2, which operate the electrical contact device when the latter is brought into action at the beginning and end of each experimental run. These pins are electrically insulated from the pendulum rod by means of keramot washers.

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(*b*) *Support*

The support for the pendulum is a truncated pyramidal iron casting, *A*, shown in fig. 2. It was designed so that its stiffness was such that the displacement of the top of the support when acted on by a horizontal force equal to the weight of the pendulum would not exceed about 0·00005 in. The calculations took no account of the internal stiffening webs, but it was found that when the displacement was subsequently determined experimentally the actual stiffness was approximately the same as the value calculated from the dimensions of the support.

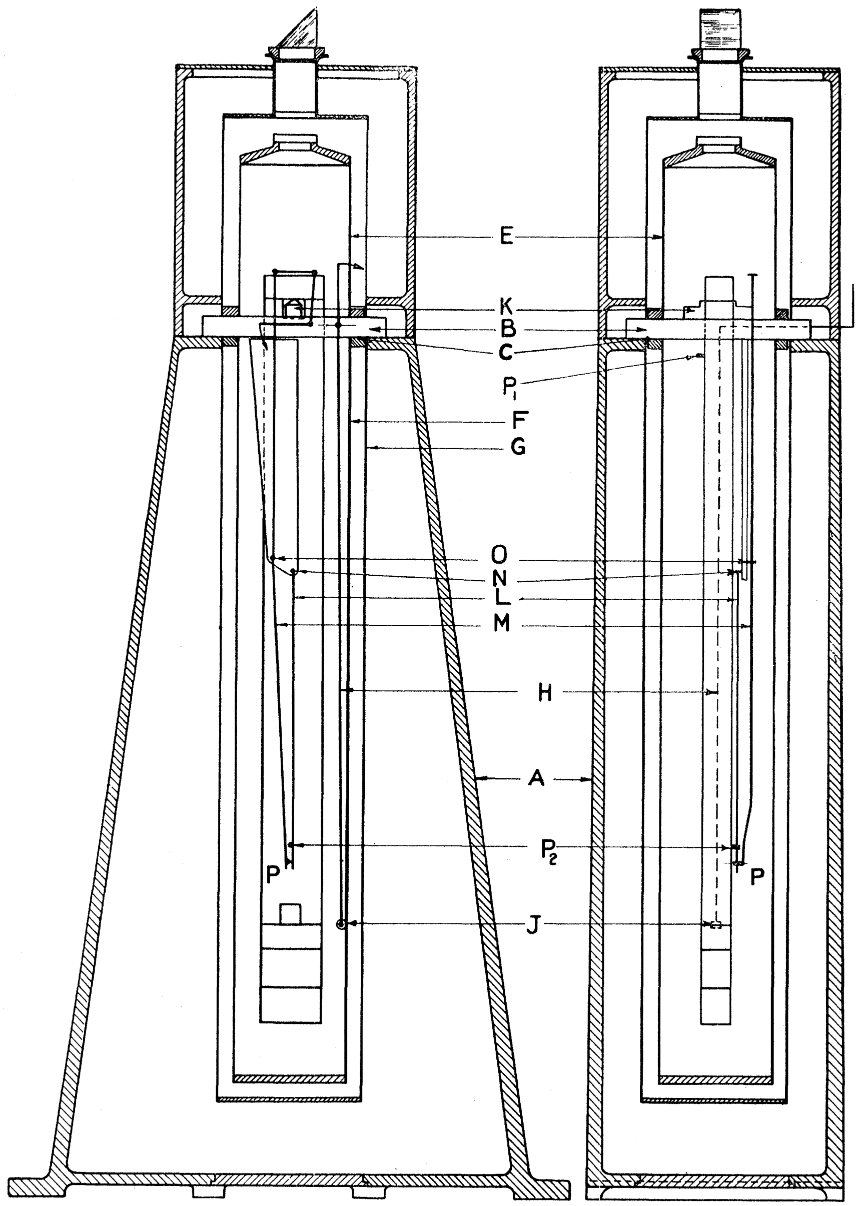

Fig. 2. Pendulum and support, diagrammatic.

The support carries the thick bronze plate B, which in turn carries the knife-edge K. The plate B carries on its under side various fittings which will be described later, whilst the upper side, in addition to the knife-edge, carries the mechanism used for lowering the pendulum carefully on to the knife-edge, together with other fittings which will also be described later.

(c) *Vacuum enclosure*

The vacuum enclosure consists of two lengths, E and F (fig. 2), of wide diameter brass tube. The upper end of the lower portion F is attached to the under side of the plate B by a vacuum-tight joint at C, whilst the lower end of the upper portion E is similarly attached to the upper side of the plate B. The vacuum enclosure surrounds the pendulum and all its subsidiary attachments, and is itself surrounded by a wider brass tube G, which separates the vacuum enclosure from the main supporting casting, the space between the latter and the outer brass tube being filled with broken cork for heat insulation purposes. The annular space between the two brass tubes contains air. The connexion between the vacuum enclosure and the pump is made by means of a hole drilled radially in the heavy brass plate B, and connecting with the enclosure on the under side of this plate. When all the vacuum joints are satisfactorily made there is no difficulty in evacuating down to a pressure of less than 1μ of mercury, and all definitive measurements of the period of the pendulum were made when the residual pressure was less than 5μ. A McLeod gauge and a Pirani gauge were used to measure the residual pressure in the system.

(d) *Subsidiary apparatus within the vacuum enclosure*

In addition to the parts already mentioned, the plate B carries the remainder of the subsidiary attachments required in connexion with the operation of the pendulum.

Three platinum resistance thermometers are used to determine the temperatures of different parts of the pendulum; two of these are supported by a rod which extends downwards from the under side of the plate B, and the third is supported by a shorter rod extending upwards from the plate B. The twelve leads from these thermometers are brought out through radial holes bored in the plate B, the airtight joints being made by passing the leads through small rubber bungs fitted into the ends of these holes and coated with sealing wax.

The starting lever, H in fig. 2, used for setting the pendulum into motion, extends downwards from the plate B, and is operated by a handle attached to a rod which passes through a hole bored in the plate B. This rod operates through an oil vacuum seal, and its handle operates between stops to prevent excessive movement of the lower end of the rod J, where it comes into contact with the pendulum. The handle also carries a counterweight so that when the starting lever is not in use, the lower end of the rod is held well out of the path of the swinging pendulum.

The electrical contact device for taking signals from the pendulum is also supported from the under surface of the plate B. It consists of a light rigid tube of aluminium L carrying at its lower end a short length of platinum wire which makes contact with another platinum wire carried on the rod M. The axes of these platinum wires are mutually perpendicular. The upper end of L is pivoted at N on the lower end of a long bracket depending from the under surface of the plate B. The rod M is also supported by this bracket, being pivoted at O near the pivot N of the aluminium tube L which carries the platinum contact, and its upper end is free to move between stops. One stop is fixed, and when the position of the rod is controlled by this fixed stop, the contact device is thrown out of action, the rod L being held in a position such that it does not come into contact with the pins attached to the pendulum (P_1 and P_2 in fig. 1) when the latter is swinging. The other stop is provided with a fine adjustment, and controls the position of M so that the platinum wire at its lower end just touches the platinum contact at the lower end of L when the pendulum is vertically at rest with L in contact with the pin P_2. The two lead-in wires serving these contacts are sealed into holes drilled radially into the plate B.

The mechanism for lowering the pendulum on to the knife-edge is shown in fig. 3. The two pivoted fingers A are connected together at the rear by a connecting bar which may be depressed by means of a micrometer screw M carried by a bracket C attached to the plate B (fig. 3). When this screw is depressed, the forward ends of the fingers

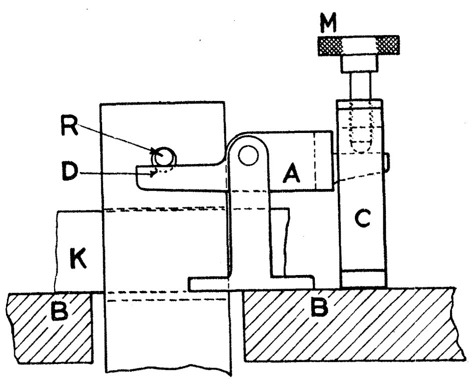

FIG. 3. Pendulum lifting device.

lift a rod R which is loosely threaded through a hole D bored through the bob of the pendulum in the direction perpendicular to the knife-edge. The pendulum hangs from this rod until it is ready to be lowered on to the knife-edge K. The following procedure was adopted to enable the pendulum to be lowered on to the knife-edge with the minimum risk of damage to the latter. The pendulum, hanging temporarily from the transverse rod supported on the lowering mechanism, was first lowered, by raising the screw M, on to two strips of tinfoil placed one near each end of the knife-edge. These strips then carried the weight of the pendulum and yielded under the residual transverse oscillations of the pendulum in the plane perpendicular to the knife-edge. When these oscillations had died out, the pendulum was raised very slightly and the tinfoil withdrawn. The pendulum could then be relowered on to the knife-edge without damaging the latter, and then the supporting rod R could be carefully withdrawn, leaving the pendulum free to swing on the knife-edge.

(e) *External subsidiary apparatus*

The circuit used for amplifying the contact signals is shown in fig. 4. The closing of the contact circuit short-circuits part of the grid bias battery, the negative grid bias being suddenly reduced from about -20 V (when practically no anode current flows) to about -5 V, when the anode current increases to between 5 and 10 mA, which is sufficient to operate the chronograph marker. This form of relay, which amplifies an extremely small current passing between the contacts, is due to Tomlinson (1922) and is quite free from trouble due to sparking which often occurs between contacts making and breaking larger currents.

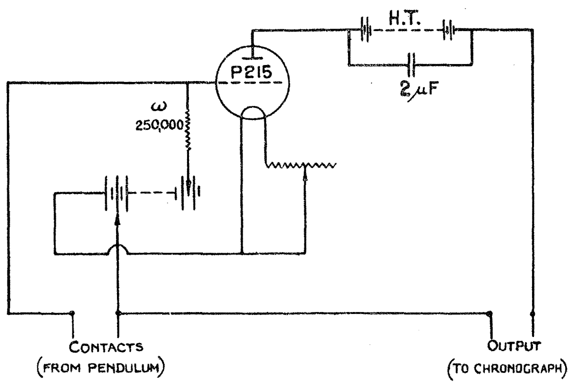

FIG. 4. Contact signal amplifying circuit.

The chronograph itself has been described by Sears and Tomlinson (1931). One of the markers records the seconds signals of one of the N.P.L. quartz crystal clocks (Essen 1936)—and the other marker records the making and breaking of the contact of the gravity pendulum with the contact device.

The approximate equality of the make/break and break/make periods was used as the criterion for the correct symmetrical setting of the contact carried on the rod M. The setting of the fine adjustment screw was varied until the successive half-periods were very closely equal, as shown by the chronograph record. When this was achieved, the two platinum wires forming the contact were just touching when the pendulum was vertically at rest. A small correction had usually to be applied to the chronograph reading owing to the fact that it was impossible precisely to equalize the two half-periods by means of the adjusting screw. This matter will be dealt with more fully in § 6, where a typical set of chronograph observations is discussed.

(*f*) *Knife-edges*

The knife-edges used throughout these experiments were made of hardened steel.* The form of the knife-edge is shown in fig. 5. The base, of area $4\frac{1}{4} \times 1$ in., was ground and lapped flat and the two sides were ground perpendicular to the base. The lapped base provided a satisfactory rigid seating when the knife-edge was screwed down on to the flat top of the massive bronze plate B (fig. 2). The parallel sides provided means for setting the pendulum symmetrically on the knife-edge, gauges being used to secure equality of clearance between the sides of the knife-edge and the sides of the cut-away portions of the pendulum rod, through which the knife-edge was inserted. The edge itself was formed at the intersection of two lapped planes making an angle of 120°, and was $2\frac{1}{4}$ in. long. Although so-called "sharp" knife-edges, formed by merely lapping two intersecting planes, were used in all the definitive measurements of the period of the pendulum, experiments were also made with knife-edges having finite radii formed by lapping the sharp edge with a rolling action. In this way "knife-edges" having cylindrical edges of radius as large as 0·28 mm. were made. The radii of all of the so-called "sharp" knife-edges were less than 20μ (0·0008 in.).

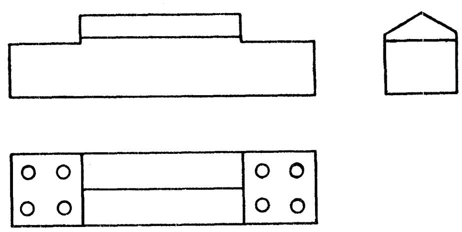

FIG. 5. Knife-edge.

3. Effect of finite radius of curvature of knife-edge on period of pendulum

The theory of a reversible pendulum swinging on a fixed knife-edge of finite radius of curvature is given in Appendix I. It is shown that the purely geometrical effect of the finite radius of curvature of the knife-edge cancels out when the pendulum is inverted, so long as the period T_1 of the pendulum when upright is closely equal to the period T_2 when the pendulum is inverted, and that the usual formulae for calculating the "computed period" T and g are applicable.

The formulae are

$$T^2 = (T_1^2 + T_2^2)/2 + (T_1^2 - T_2^2)\,(h_1 + h_2)/2(h_1 - h_2),$$

and

$$gT^2 = (h_1 + h_2)\,\pi^2.$$

h_1 and h_2 are the distances of the centre of gravity of the pendulum from the knife-edge when the pendulum is upright and inverted, respectively, and their values need only be known approximately for calculating the small term $(T_1^2 - T_2^2)\,(h_1 + h_2)/2(h_1 - h_2)$ in the expression for T.

* Brinell hardness no. 590–610.

(h_1+h_2), in the expression for g, must, however, be determined with the highest possible precision. It is the distance between the centre of rotation of the pendulum when oscillating in the upright position and the centre of rotation when the pendulum is inverted. Owing to the compression which occurs at the contact between the knife-edge and the bearing plane, (h_1+h_2) is not exactly equal to the distance between the bearing planes. The latter distance must be corrected by the amount $+2\alpha$, where α is the compression occurring at the knife-edge.

The computation of this correction 2α is given in Appendix II; its value, if both the bearing surface and the knife-edge had been of steel, is shown to be less than $0{\cdot}5\mu$ for any radius of knife-edge between 2·5 and 25μ. This corresponds with a correction to the measured value of g amounting to not more than 5 parts in 10 millions, but since, in the actual pendulum used, no definite information was available regarding the elastic constants of the chromium plated planes supported by the steel knife-edge, the calculated values are not very reliable. The calculated correction, however, is small, and since the modulus of rigidity of the chromium plating is almost certainly greater than that of steel, the true value of the correction is probably still smaller, and might in fact have been neglected in computing the final value of g.

The actual radii of the knife-edges were measured by an adaptation of the method described by Schmerwitz (1932). The knife-edge was mounted horizontally on a bracket attached to a horizontal spindle which could be smoothly rotated in a bearing and the angle of rotation could be measured by means of a micrometer attached to the end of a 10 in. arm fixed radially to the spindle. An accurately made steel beam, of width equal to the width of the pendulum in contact with the knife-edge, and of section as shown in fig. 6 was carefully balanced on the knife-edge. The under surface of this beam, where it came into contact with the knife-edge, was ground and lapped accurately flat and the upper surface was lapped flat so as to act as a plane mirror. Then, if the knife-edge is rotated through an angle θ, the beam tilts through the angle ϕ, where $r = x \tan \phi/(\theta-\phi)$. r is the radius of curvature of the knife-edge, and x is the distance of the centre of gravity of the beam below the point of contact. x was calculated from the dimensions of the beam, and ϕ was determined by means of an auto-collimating system, the image of a distant cross-wire formed by reflexion in the flat polished upper surface of the beam being observed. The focal length of the auto-collimating lens was 160 cm., so that a movement of the image of 1 mm. on the scale was equivalent to a tilting of the beam of 0·00031 radian.

θ, which need not be known so accurately as ϕ, can easily be read to 0·0001 radian. Two beams were required to cover the range of radii used, one for the knife-edges of "formed" radius of 75μ and more, and the other for the smaller formed radii and for the so-called "sharp" knife-edges having radii of less than 25μ. In the former beam, x was 1·39 mm. and in the latter x was 0·0645 mm.

In all cases the measurement of radius was confined to small values of ϕ, corresponding approximately with the small displacement of the pendulum itself when swinging on the knife-edge. The smaller radii (up to $r = 25\,\mu$) determined by this method showed fairly large variations when measured under different conditions. These variations were attributed to the slight residual magnetism remaining in the knife-edges and steel beams, which could not be completely demagnetized after they had been lapped. The mean values of the radii accepted for these "sharp" knife-edges were however sufficiently accurate, since radii up to 15μ only affected the apparent value of g by about one part in a million. The larger radii (up to $r = 3$ mm.) are probably accurate to within 5%. This accuracy is sufficient, since the value of r only enters into small corrections. In the case of the "sharp" knife-edges, measurements of r before and after swinging the pendulum showed that all newly lapped "sharp" knife-edges had approximately the same radius of curvature, about $2{\cdot}5\mu$, but after the knife-edge had been used under the pendulum its radius of curvature sometimes increased to 10μ or even more, the whole of the increase usually occurring the first time the pendulum was swung on the knife-edge. In other cases, however, as will be

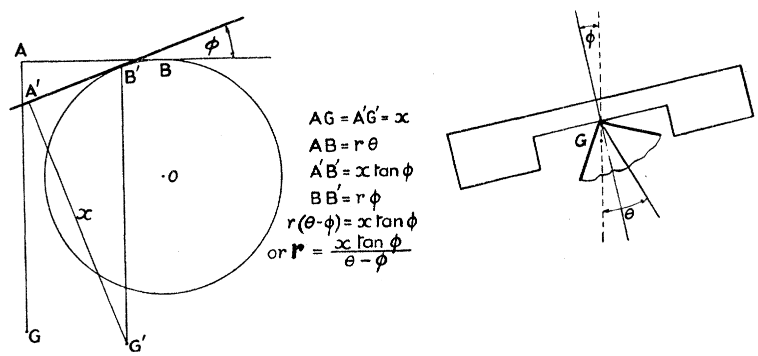

FIG. 6. Determination of radius of knife-edge.

seen from column 3 of Table XI, the radius obtained was approximately $2{\cdot}5\mu$, and it remained at closely $2{\cdot}5\mu$ even after the knife-edge had been used under the pendulum.

The effective knife-edge radius was in every case taken to be the value measured after the knife-edge had been used under the pendulum.

Although the purely geometrical effect of a small finite radius of curvature of knife-edge is shown in Appendix I to have a negligible effect on the value of g so long as T_1 and T_2 are closely equal, it was desirable to investigate the possibility of an irreversible loss of energy occurring when knife-edges of larger radius are used. For this purpose, the knife-edges having "formed" radii of values up to $r = 0{\cdot}28$ mm. were used, and the periods T_1 and T_2* were determined for the various values of r. The results are given in Table I.

* T_1, T_2 and T are strictly the half-periods; but since a pendulum approximately 1 m. in length is usually referred to as a "seconds pendulum", the half-period has been described throughout as the "period".

Table I. Effect of radius of knife-edge on period

1	2	3	4	5	6	7
Radius r μ	Period T_1 sec.	$\Delta T_1\times 10^7$ sec.	Period T_2 sec.	$\Delta T_2\times 10^7$ sec.	g cm./sec.^{2}	$\Delta g\times 10^7$
11	1·002 8875	± 5	1·002 8784	± 3	981·1791	±12
77·5	8361	± 6	7387	±11	·1857	±25
173	7744	±13	5668	±11	·1891	±34
282	7057	±11	3815	±24	·1995	±53

Each value of T_1 and T_2 given in the table is the mean of five separate measurements of period, the mean residuals being given under ΔT_1 and ΔT_2. The apparent value of g is calculated for each set of measurements and the residuals Δg are calculated from the formula $\Delta g = 2\surd\{(\Delta T_1)^2+(\Delta T_2)^2\}$.

The values of g are calculated by means of the formulae of Appendix I, viz.

$$T^2 = (T_1^2+T_2^2)/2+(T_1^2-T_2^2)(h_1+h_2)/2(h_1-h_2),$$

and $$gT^2 = \pi^2(h_1+h_2),$$

where h_1, h_2 and (h_1+h_2), the corrected length of the equivalent simple pendulum, are taken from §§ 9 and 10.

The apparent values of g are found to be related to the values of r by the equation

$$g_r = 981\cdot 1795[1+0\cdot 65r\times 10^{-7}],$$

where r is the radius of curvature of the knife-edge in μ, and are given in column 6 of the table.

It is evident that the effect of the radius of curvature of the "sharp" knife-edges on the apparent value of g is small, the rate of change of g with r being about 13 parts in 10^7 for an increase in radius of 20μ. The definitive values of g, all determined with "sharp" knife-edges of radii less than 20μ, were therefore obtained by reducing the calculated values by extrapolating to zero radius in accordance with the above equation.

The apparent increase in g with increasing r must be due to an irreversible loss of energy by the friction between the knife-edge and plane. Even with planes of hard material in contact with steel knife-edges, the radius of curvature of the latter must be kept below 15μ if g is to be known to one part in a million.

4. Effect of elasticity of support on period of pendulum

Two methods were considered for determining the effect of the elasticity of the pendulum support on the period of the pendulum.

The simple theory, generally given in text-books,* shows that the elasticity of a pendulum support has the effect of increasing the apparent length of the pendulum

* See, e.g., Poynting and Thomson (1905).

by an amount equal to the distance by which the support would be displaced by a force equal to the weight of the pendulum applied horizontally at the point of suspension of the pendulum. The pendulum support was therefore designed of such dimensions that a horizontal force of the order of 20 lb., equal to the weight of the pendulum, would not be expected to displace the top of the support by more than about 0·00005 in. The calculations of the rigidity of the pendulum support could only be made approximately, and no allowance was made for the stiffness of the internal webs on the support which would have the effect of making it more rigid. Further, no allowance was made for the rigidity of the concrete pillar on which the pendulum support rested. It was therefore considered desirable not to accept the calculated value of this correction, but to determine the correction experimentally.

Schumann (1899) has investigated theoretically the motion of two similar pendulums of equal periods, swinging side by side *in vacuo* on the same support, and has shown that the correction to the period due to the elasticity of the support could be calculated

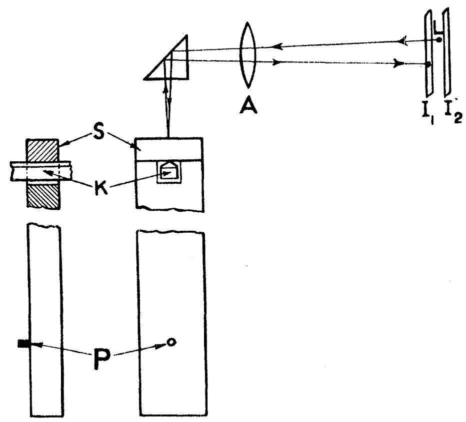

Fig. 7. Subsidiary pendulums for determining flexure of support.

from simultaneous observations of the amplitudes of the two pendulums. This method, therefore, was used to determine the actual correction to be applied to the observed period due to the elasticity of the support right up to and including the knife-edge itself.

For this purpose two subsidiary pendulums of simple design, having the general form of rectangular bars of brass, $47 \times 2 \times \frac{3}{4}$ in., were made. The weight of each was approximately the same as the weight of the reversible pendulum itself. One of these subsidiary pendulums is shown in fig. 7. Each is provided with a pin, P, to operate the contact device, and the upper end is cut away to permit the insertion of the knife-edge, K. To the upper end of each is attached a rectangular block of steel, S, case-hardened, of the same cross section as the bar, and $\frac{3}{8}$ in. thick. The upper and lower surfaces of this block are ground and lapped flat, the lower surface to provide a plane bearing surface to rest on the knife-edge, and the upper to act as a plane mirror to enable the amplitude of swing of the pendulum to be observed. The periods of each of these pendulums were determined, and by removing metal from the lower end of one of them, their lengths were so adjusted that their periods were equal to within 10^{-5} sec., the half-period of each being $0{\cdot}89504_5$ sec.

The two pendulums were supported side by side on the same knife-edge, and both were brought to rest. One of the pendulums was now set swinging. After a lapse of time t, the amplitude of this driving pendulum will have decreased to ϕ and the amplitude of the driven pendulum will have increased from zero to ψ. It has been shown by Schumann (1899) that if α is the apparent lengthening of either of the pendulums, due to the elasticity of the support, then $\alpha = (\psi/t\phi)\,(2gT^3/\pi^3)$, where T is the half-period of either of the pendulums.

The amplitudes ϕ and ψ were determined by observing the images of the horizontal filament of a 12 V motor headlamp L, formed by an autocollimating lens A of focal length about 180 cm. and reflected from the upper polished surfaces of the steel blocks at the top of each pendulum. The blocks were made very slightly wedge-shaped so that the two images were formed side by side on two vertical scales, I_1 and I_2, placed about 2 in. apart.

A typical series of results is plotted in fig. 8. The amplitudes ϕ and ψ are given in millimetres measured on the fixed scales I_1 and I_2, and the time in minutes elapsing from the moment when the driving pendulum is given its initial motion is plotted horizontally. The values given in the following Table II are taken from the rising portion of the ψ-curve, ignoring the part near the origin, where the value of ψ is small and therefore the errors of observation may be proportionally large,* and avoiding the portion of the curve where ψ is approaching a maximum; Schumann's formula does not apply to this portion of the curve.

TABLE II

Residual pressure in vacuum enclosure: $0{\cdot}1\mu$.
Initial $\phi = 128{\cdot}5$ mm. at 2.50 p.m.

Time	t (in sec.)	ψ	θ	$\psi/t\phi \times 10^6$	(Obs.–Mean)
3·30	2400	$0{\cdot}9_5$	124·5	3·17	−0·03
3·45	3300	$1{\cdot}3_5$	123·5	3·31	+0·11
4·00	4200	1·7	122	3·32	+0·02
4·15	5100	2·0	121	3·24	+0·04
4·30	6000	$2{\cdot}2_5$	119·5	3·14	−0·07
4·45	6900	$2{\cdot}4_5$	118·5	3·00	−0·20
			Means	3·20	±0·08

The calculation of the expression for α depends on the assumption that the transfer of energy from the driving to the driven pendulum only takes place through the non-rigid support—in other words, that the pendulums are swinging in a vacuum. To

* The observations of ψ near the origin are further complicated by the fact that it was almost impossible to start with the driven pendulum absolutely at rest. The initial amplitude was, however, usually less than $\pm 0{\cdot}1$ mm. and the initial energy of the pendulum may consequently be assumed negligible.

determine the effect of the residual pressure in the vacuum enclosure, experiments were carried out with various residual pressures inside the vacuum enclosure. The resulting values of $\psi/t\phi$ for various pressures are given in the first two columns of Table III.

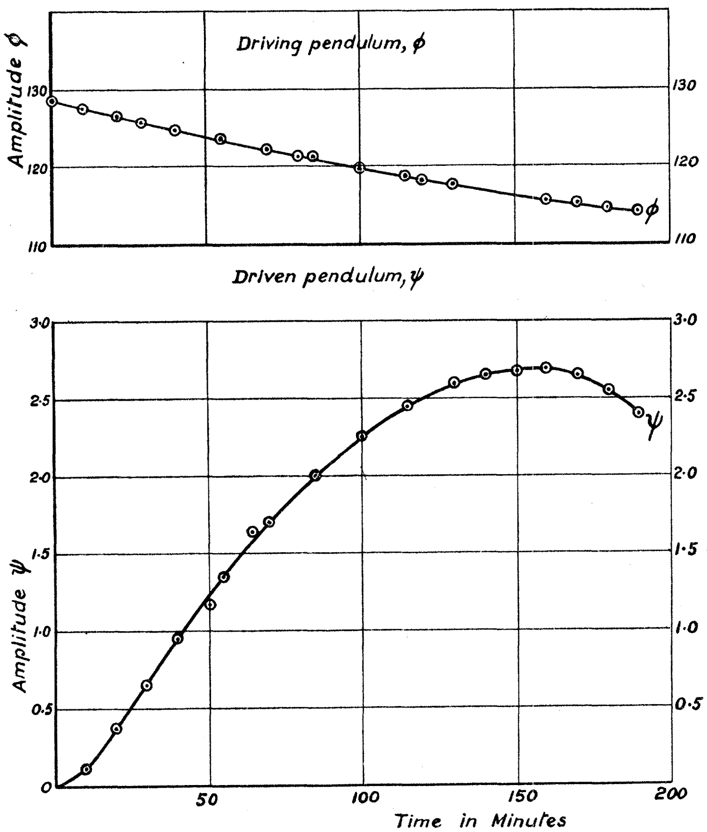

FIG. 8. Flexure of pendulum support; relations between ϕ, ψ and t.

TABLE III

1 Pressures p (in μ of Hg)	2 $\psi/t\phi \times 10^6 \equiv R$ (from observations)	3 R (calculated from parabolic formula)	4 Residuals $R_{obs.} - R_{calc.}$
$<0{\cdot}1$	3·00	(3·02 for $p = 0{\cdot}05\mu$)	($-0{\cdot}02$)
0·1	3·20	3·07	$+0{\cdot}13$
0·4	3·34	3·26	$+0{\cdot}08$
0·8	3·37	3·40	$-0{\cdot}03$
2·8	3·68	3·73	$-0{\cdot}05$
15	4·23	4·33	$-0{\cdot}10$
61	4·97	4·97	$\pm 0{\cdot}00$
12×10^3	8·48	8·47	$+0{\cdot}01$
		Mean	$\pm 0{\cdot}06$

It is obvious that the apparent value of α increases rapidly as the pressure is increased from the lowest obtainable in the system (*ca.* $0{\cdot}1\mu$) up to 12 mm. of Hg. If the above values of $\psi/t\phi \times 10^6$, which for convenience may be called R, are plotted against the values of $\log_{10} p$, a smooth curve is obtained, approximating to a parabola with its axis vertical, from which the limiting value of R, and therefore of α, when the pressure is zero, may be derived.

The results are plotted in fig. 9, and the parabola derived therefrom, and used to calculate the values of R given in the third column of Table III is

$$(\log_{10} p + 1{\cdot}65)^2 = 6(R - 3{\cdot}0).$$

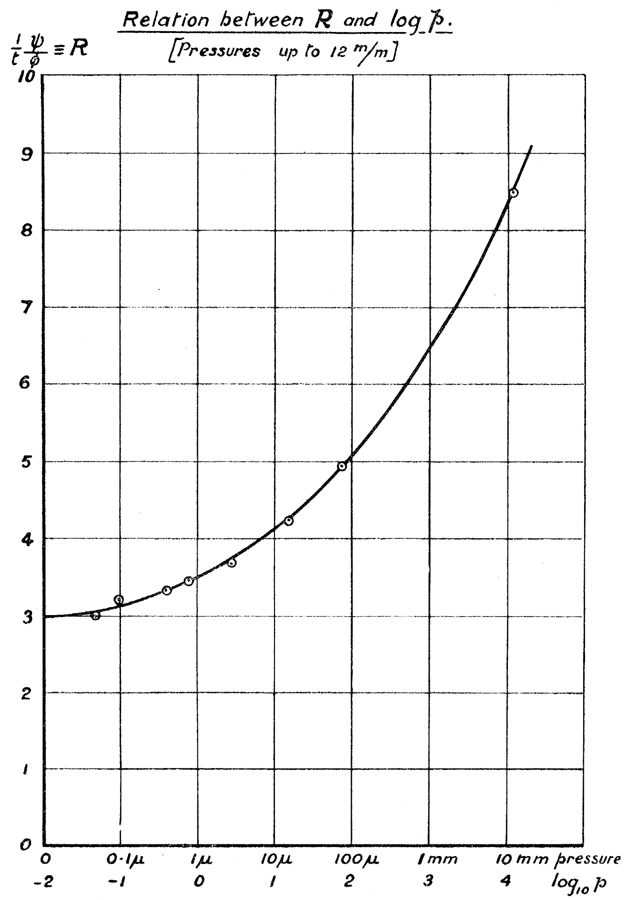

FIG. 9. Flexure of pendulum support; relation between R and $\log p$ (pressures up to 12 mm.).

From this, the limiting value of R is 3·0, with which is associated an uncertainty of about $\pm 0{\cdot}1$, since it depends largely on the accuracy of the observations at the lower pressures.

The values of the residuals $R_{\text{obs.}} - R_{\text{calc.}}$ given in the last column of the table are of the same order of magnitude as the experimental residuals given in the last column of Table II. Hence the values of R predicted by the parabola are at least as accurate as the experimental values.

Accepting $R \equiv \psi/t\phi \times 10^6 = 3{\cdot}0 \pm 0{\cdot}1$ for zero pressure, the value of

$$\alpha = (\psi/t\phi)\,(2gT^3/\pi^3) = 1{\cdot}3_6\,\mu,$$

and since the weight of each of the subsidiary pendulums was about 9·7 kg. as compared with 10·4 kg. for the reversible pendulum, the correction to be added to the measured length of the reversible pendulum, due to the elasticity of its support, may be taken as $1{\cdot}5\mu$. In addition to varying the pressure in the enclosure during these experiments the value of the initial amplitude varied from 0·014 radian (51 mm. on scale) to 0·036 radian (129 mm. on scale), and the observed values of R confirmed that the value of the expression $\alpha = (\psi/t\phi)\,(2gT^3/\pi^3)$ is independent of the initial amplitude of the driving pendulum within this range.

5. Details of experimental procedure

The whole of the internal subsidiary fittings—two thermometers, contact device, and starting lever—attached to the under side of the plate B (fig. 2), had of course to be assembled before the plate was fixed to the flanged top of the vacuum enclosure. For this purpose a bracket attached to the wall of the room was used temporarily to carry this plate whilst the under parts were being assembled. The top plate assembly, complete except for the knife-edge, was then lowered into the vacuum enclosure, and the whole was transferred to the pendulum support, the under side of the periphery of the plate B resting on the flat top of the pendulum support, in which was cut a circular hole just large enough to admit the large tube forming the vacuum enclosure. The periphery of the plate was held down by means of six large studs screwed into the top of the support.

The support was originally set up so that its top surface was horizontal, and since the plate B was ground so as to make its top and bottom surfaces parallel, the upper surface of the plate B when in position was also horizontal. The upper surface has to carry the knife-edge which must be horizontal. The remainder of the subsidiary fittings—the third thermometer and the device for lowering the pendulum on to the knife-edge—were now assembled on the top of the plate B.

The pin, R, shown in section in fig. 3, was now passed through the transverse hole D in the upper bob of the pendulum, and by means of a double hook engaging with the free ends of the pin, the pendulum was carefully lowered through the rectangular hole in the plate until the pin rested on the two fingers A of the pendulum lifting device (fig. 3), the micrometer screw controlling the height of these fingers having previously been adjusted so that the knife-edge could be inserted through the square cut-away portion in the pendulum without bringing the edge into contact with the chromium-plated bearing plane. The clearance was about a millimetre. The hooks were now removed, and the pendulum remained suspended by the pin resting on the two fingers. The knife-edge was then firmly fixed by means of four steel screws at each end, and the pendulum adjusted slightly along the pin until the clearances between the sides of the knife-edge block and the sides of the cut-away portion of the pendulum were equal. This cut-away portion was symmetrical with the axis of the pendulum; hence, when the clearances were adjusted to equality the knife-edge was vertically above the centre of figure of the pendulum to within about 30μ. The pendulum was also set symmetrically in the direction of the axis of the knife-edge, so that, when at rest, it always occupied the same position relative to the rectangular hole in the top plate and to the fittings surrounding it. The cylindrical gauges used to set the pendulum symmetrically on the knife-edge could now be withdrawn, and, when all the movements of the pendulum had died out, the pendulum could be carefully lowered on to the knife-edge by means of the micrometer screw. The pin which had previously taken the weight of the pendulum could then be removed.

The next operation was to adjust the position of the arm M (fig. 2) carrying the contact pin P so that, when the pendulum was vertically at rest, the pin on the pendulum was just in contact with the pin P. This was done by trial and error adjustment of the micrometer screw (controlling the position of the pin P) until the times of successive half-periods of the pendulum, as recorded on the chronograph, were as nearly as possible equal.

The cover, E, of the vacuum enclosure was now placed in position. The vacuum-tight seal consisted of a thin thread of soft wax which became pressed into a thin film between the top plate B and the flange of the cover. The cover E was provided with a plate glass window at the top, also sealed on with soft wax. The vacuum enclosure was then exhausted by means of a mercury vapour pump backed by a rotary oil pump.

The apparatus was left overnight to attain steady temperature conditions, the room being thermostatically controlled at about 20° C, and resistance heaters being provided between the base of the pendulum support and the top of the concrete pillar to counteract the tendency to vertical temperature gradients due to heat losses through the concrete pillar, which extended some 5 feet down to earth below floor level.

6. Measurement of period of pendulum

As soon as steady temperature conditions had been attained and the pressure within the pendulum enclosure had been reduced to below 1μ of mercury, the pendulum could be started in oscillation by means of the lever H (fig. 2). For all definitive measurements of period, the initial amplitude was not allowed to exceed 0·004 radian in order that the amplitude correction should not exceed one part in a million. The contact device was then brought into action, the controlling lever M being in contact with the micrometer stop. After the pendulum had been swinging for at least half a minute, the controlling lever was swung out of action. This was done in a systematic manner whilst observing a chronometer, so that the time at which the pendulum ceased recording contact makes and breaks and became a free pendulum was known to within a quarter of a second.

After the pendulum had been swinging freely for about 3 hr., the contact device was again brought into action, at a time known to within a quarter of a second by

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the chronometer, and, after about half a minute of record had been obtained on the chronograph, the experiment ended. In the meanwhile, during the course of the 3 hr. interval, measurements of temperature had been made with the platinum resistance thermometers, and the amplitude had been continuously observed.

In order to understand how the uncorrected period of the pendulum is derived from the chronograph records obtained in the way described above, it is desirable to consider a typical series of measurements of a record. Table IV gives a complete schedule of all the measurements of two records required to calculate a single value of the uncorrected period of the pendulum.

The scale of the chronograph records is 1 division per 0·01 sec., and the positions of the signals on the record were read to the nearest 0·05 division, i.e. to $\frac{1}{2}$ msec.

The upper half of the table gives the chronograph readings from which the initial fiducial signal of the timing interval is derived and the lower half those from which the final fiducial signal is derived. Column 1 gives the readings for the signals from the standard clock; only the two marked with asterisks are required for the calculation of the period. The second column gives the readings from the "make" and "break" signals corresponding with successive passages of the pin on the pendulum through the position occupied by the pin P on the contact rod M when the latter was at rest against the micrometer stop. The last ten or twelve signals preceding the removal of the contact device were observed, and of these signals the last had to be ignored, as it was liable to be disturbed by the movement of the lever controlling the position of the contact device, which movement occupied about $\frac{1}{2}$ sec. From the differences given in column 3, however, it was possible to calculate the unperturbed position of the last make or break before the contact ceased to operate, i.e. the commencement of the timing interval. The procedure was as follows. Successive values of the differences "make *minus* break" and "break *minus* make" were alternately longer and shorter than the mean difference which would have been observed if the stop on the contact device had been set exactly below the centre of gravity of the pendulum when at rest in a vertical position. The observed differences (in units of 0·01 sec.) are given under 4δ in column 4, from which δ, the correction which should be added to or subtracted from the observed signal, is derived, and the unperturbed readings in column 2 are extrapolated by means of the mean differences in column 3 and corrected by $\pm\delta$ from column 4 to derive the corrected fiducial signal given in column 5.

The mean of the differences given in column 3 is 100·195 chronograph divisions, which is a measure of the period of the pendulum when it is operating the contact device. The extrapolation of the first two readings in column 2 gives the first two corrected fiducial signals in column 5, thus

$$545{\cdot}8\ \ +12\times100{\cdot}195+0{\cdot}18 = (1)748{\cdot}32$$

and $$646{\cdot}25+11\times100{\cdot}195-0{\cdot}18 = (1)748{\cdot}22 \text{ respectively,}$$

TABLE IV

1 Seconds signals from standard clock (div.)	2 Signals from gravity pendulum (div.)	3 Differences (100 divs. to 1 sec.)	4 Correction for asymmetry (unit 0·01 sec.) 4δ	δ	5 Corrected fiducial signal (unit 0·01 sec.)	6 Residuals (unit 0·01 sec.)
540·9	545·8 B	—	—	—	748·32	+0·04
640·85	646·25 M	100·45	—	—	·22	−0·06
740·9	746·1 B	99·85	0·6	0·15	·23	−0·05
841·0	846·7 M	100·6	0·75	0·19	·27	−0·01
941·05	946·5 B	99·8	0·8	0·20	·24	−0·04
041·05	047·05 M	100·55	0·75	0·19	·24	−0·04
141·15	146·95 B	99·9	0·65	0·16	·30	+0·02
241·1	247·6 M	100·65	0·75	0·19	·39	+0·11
341·15	347·35 B	99·75	0·9	0·22	·31	+0·03
441·05	447·9 M	100·55	0·8	0·20	·31	+0·03
541·1	547·75 B	99·85	0·7	0·17	·32	+0·04
641·1	648·2 M	—	—	—	748·21	−0·07
*741·05 $=S_1$	(742) B	Mean 100·195	Mean 0·74	Mean ±0·18	Mean 748·28 $=\tau_1$	Mean ±0·05
S_2-S_1 =100·80	Interval 12230 sec.				$\tau_2-\tau_1$ =127·33	
		Mean 100·140	Mean 0·52	Mean ±0·13	Mean 875·61 $=\tau_2$	Mean ±0·06
*841·85 $=S_2$	(876) M	—	—	—	—	—
941·8	(976) B	—	—	—	—	—
041·9	076·1 M	—	—	—	875·69	+0·08
142·0	175·8 B	99·7	—	—	·51	−0·10
241·95	276·2 M	100·4	0·7	0·17	·51	−0·10
341·9	376·2 B	100·0	0·4	0·10	·63	+0·02
441·85	476·6 M	100·4	0·4	0·10	·63	+0·02
541·85	576·4 B	99·8	0·6	0·15	·55	−0·06
641·75	676·9 M	100·5	0·7	0·18	·65	+0·04
741·8	776·75 B	99·85	0·65	0·16	·62	+0·01
841·95	877·1 M	100·35	0·5	0·12	·57	−0·04
942·0	977·1 B	100·0	0·35	0·09	·69	+0·08
041·95	077·5 M	100·4	0·4	0·10	875·69	+0·08

and the mean of twelve such corrected readings, 748·28, is given in column 5 as the "corrected fiducial signal", τ_1, for the commencement of the timing interval. The mean residual, $\pm 0{\cdot}05$ division of the chronograph, given in column 6, is comparable with the accuracy with which the chronograph records were read.

The lower half of the table gives the corresponding observations and computations from which the final fiducial signal τ_2, for the conclusion of the timing interval, is derived.

Now, the time interval between these fiducial signals is approximately known (to the nearest half-second) by the chronometer readings corresponding with the operations of bringing the contact device out of action and subsequently replacing it. The true time interval between the fiducial signals is then derived from

$$(\tau_2-\tau_1)-(S_2-S_1) = 127{\cdot}33-100{\cdot}80 = 26{\cdot}53 \text{ chronograph divisions,}$$
$$= 0{\cdot}2653 \text{ sec.}$$

Only the fractional part of a second is determined by this calculation since of course the chronograph is stopped during most of the timing interval. The whole number of seconds is known from the chronometer readings, which determine it to the nearest half-second. In most cases there was no difficulty in obtaining the correct whole number, but occasionally, where the fractional part happened to be very closely 0·5 sec., the two most likely values were used and two values of the period were derived. When the timing interval was of the order of 3 hr. (i.e. when the number of swings of the pendulum exceeded 10,000) the two values of the period thus obtained did not differ from their mean by more than 1 part in 10^7. Actually an additional check on the assignment of the correct whole number of seconds was provided by a signal from another clock which gave a record of every tenth second on the chronograph, but the readings of this signal are, for simplicity, omitted from the above table.

The time interval between τ_1 and τ_2 in the case under consideration (Table IV) is 12230·2653 sec., there being no uncertainty in assigning the correct whole number of seconds.

Now, by a series of similar but shorter experimental runs made in the early stages of the work, the period of the pendulum was determined, by successive approximations, to be 1·00289 sec. to the nearest 10^{-5} sec. Thus runs of the order of 50–100 sec. gave the period of 1·003 correct to within 10^{-3} sec. and runs of 1000 sec. gave the period 1·0029 correct to within 10^{-4} sec., the closer approximation, 1·00289 sec. being confirmed by many experimental runs of between 1000 and 5000 sec.

The approximate value of n (the number of swings) in the experimental run analysed in Table IV is therefore

$$12230{\cdot}2653/1{\cdot}0029 = 12194{\cdot}9 \quad \text{or} \quad 12230{\cdot}2653/1{\cdot}00289 = 12195{\cdot}02,$$

and since n must be an integer, the period must be

$$12230{\cdot}2653/12195 = 1{\cdot}002{,}891{,}8 \text{ sec.*}$$

The uncorrected period of the pendulum is thus known in terms of the quartz crystal clock. The rate of the latter is known to remain constant to well within 1 part in 10^8 during periods of a few hours, and its mean daily rate is determined by comparison with three sets of astronomical time signals (wireless time signals from Greenwich, Hamburg and Paris Observatories, radiated by Rugby, Nauen and Paris respectively). The actual corrections to the period due to the rate of the quartz crystal clock ranged from -3×10^{-7} to $+12\times10^{-7}$ sec. during the whole of the time occupied by the definitive measurements of period, and the error introduced by uncertainty in clock corrections is quite negligible.

* This is strictly the half-period; but throughout this work it has been convenient to refer to the time between two successive passages of the pendulum through the vertical as the "period" of the pendulum, numerically as well as symbolically.

It will be seen from column 3 of Table IV that the period of the pendulum whilst the contact rod was in operation at the beginning of the experiment was $1\cdot0019_5$ sec., and at the end of the experiment 1·0014 sec., the period of the "free" pendulum being 1·0029 sec. The reduction in period due to the action of the contact rod depends on the amplitude, being about 0·001 sec. when the amplitude was 0·0035 radian and 0·0015 sec. when the amplitude had decreased to 0·00265 radian. The corresponding figures obtained with the pendulum inverted were 0·0026 sec. with amplitude 0·0042 radian and 0·0042 sec. with amplitude 0·0025 radian. Naturally, the disturbing effect of the contact rod is greater when the pendulum is inverted, since the centre of gravity of the pendulum is then raised. It is evident that, since the contact rod cannot be brought into action and withdrawn from action instantaneously, the measured timing interval may be in error by some fraction of 7 msec. in the worst case (probably not more than 4 msec. generally) owing to the disturbing effect of the contact rod acting for a fraction (not more than a half) of a second. Since the timing interval in all the definitive measurements of period exceeded 10,000 sec., the possible error in period due to this cause should not exceed 4×10^{-7} sec., but variations of this order are to be expected, particularly in measurements of the period T_2, when the pendulum is inverted. This possibility will receive special attention in § 7 (*b*) where, in investigation of the amplitude correction, it is necessary to use very small amplitudes. In such cases the timing interval was considerably increased in order to reduce the effect of the contact rod on the period of the pendulum (see Tables V and VI).

7. Corrections to period of pendulum

The individual periods obtained by analysis of the chronograph records have to be corrected for temperature and reduced to their limiting values for zero amplitude. Since all the measurements were made at or near the temperature of 20° C, it was convenient to regard 20° C as the standard temperature and to reduce all measurements of length and period to that temperature.

(*a*) *Temperature corrections*

The long platinum thermometer determines the mean temperature of the pendulum rod and the two short ones give the temperatures of the two bobs. Let θ_0 be the difference between the temperature of the rod and the standard temperature, 20° C, to which all measurements of length and period are reduced, and let θ_1 and θ_2 be the corresponding values for the upper and lower bobs respectively. Let α_0 be the coefficient of linear expansion of the Y-alloy rod and α_1 that of the delta metal bobs. Let h_0 and h_1 be the distances of the centres of gravity of the pendulum rod and upper bob, respectively, from the knife-edge, and let k_0 and k_1 be the radii of gyration of the pendulum rod and of the upper bob about the knife-edge. Let k_2 be the radius of gyration of the lower bob about its own centre of gravity and let l be the half-length of the lower bob. m_0, m_1 and m_2 are the masses of the rod and bobs respectively.

Then the period of the pendulum at the standard temperature, 20° C, is given by

$$\frac{gT_{20}^2}{\pi^2}=\frac{m_0k_0^2+m_1k_1^2+m_2k_2^2+m_2(2h_0+l)^2}{m_0h_0+m_1h_1+m_2(2h_0+l)},$$

which may be written $gT_{20}^2/\pi^2 \equiv A/B$.

The period T_θ at the temperature of the observation is then given by

$$\frac{gT_\theta^2}{\pi^2}=\frac{\begin{Bmatrix}m_0k_0^2(1+2\alpha_0\theta_0)+m_1k_1^2(1+2\alpha_1\theta_1)+m_2k_2^2(1+2\alpha_1\theta_2)\\ +4m_2h_0^2(1+2\alpha_0\theta_0)+4m_2h_0l(1+\alpha_0\theta_0+\alpha_1\theta_2)+m_2l^2(1+2\alpha_1\theta_2)\end{Bmatrix}}{m_0h_0(1+\alpha_0\theta_0)+m_1h_1(1+\alpha_1\theta_1)+2m_2h_0(1+\alpha_0\theta_0)+m_2l(1+\alpha_1\theta_2)},$$

which may be written

$$\frac{gT_\theta^2}{\pi^2}\equiv\frac{A+x}{B+y}=\frac{gT_{20}^2}{\pi^2}\left(1+\frac{x}{A}-\frac{y}{B}\right)\text{ approx.},$$

where

$$x\equiv 2[m_0k_0^2\alpha_0\theta_0+m_1k_1^2\alpha_1\theta_1+m_2k_2\alpha_1\theta_2+4m_2h_0^2\alpha_0\theta_0+2m_2h_0l\alpha_0\theta_0+2m_2h_0l\alpha_1\theta_2+m_2l^2\alpha_1\theta_2]$$

and $$y\equiv m_0h_0\alpha_0\theta_0+m_1h_1\alpha_1\theta_1+2m_2h_0\alpha_0\theta_0+m_2l\alpha_1\theta_2.$$

Since x and y are both small, approximate values of A and B, determined from the mass and dimensions of the various parts of the pendulum, may be used in this formula. The period at the standard temperature of 20° C is then given by

$$T_{20}=T_\theta(1-x/2A+y/2B).$$

Substituting the values of A and B, and the appropriate constants in the expressions for x and y, derived from the measured values of the mass and dimensions of the component parts of the pendulum, the following formulae were obtained:

(*a*) pendulum upright:

$$T_\theta/T_{20}=1+[0{\cdot}037\alpha_1\theta_2+0{\cdot}46\alpha_0\theta_0+0{\cdot}002\alpha_1\theta_1];$$

(*b*) pendulum inverted:

$$T_\theta/T_{20}=1+[0{\cdot}006\alpha_1\theta_2+0{\cdot}395\alpha_0\theta_0+0{\cdot}10\alpha_1\theta_1];$$

and, substituting the values of the coefficients of expansion α_0 and α_1, for various temperatures near 20° C, a table of temperature corrections was drawn up.

The coefficient of linear expansion α_0 of the Y-alloy rod was determined by direct measurements of its length at various temperatures between 1 and 31° C in terms of a standard scale, by means of a comparator. For this purpose, the two pins P_1 and P_2 in fig. 1 were temporarily replaced by plugs, each polished at one end, and each bearing a fine fiducial line engraved on its polished surface. The distance between these lines (approximately 78 cm.) was measured at various temperatures, and from the results the following equation was derived:

$$L_\theta = L_{20}[1+21\cdot94(\theta-20)\times10^{-6}+15\cdot6(\theta-20)^2\times10^{-9}].$$

The coefficient of linear expansion α_1 of the delta metal bobs was determined by measurements of one of the chromium plated blocks, B or C in fig. 1. By wringing a half-silvered glass plate to each of the flat chromium plated surfaces so as to project beyond the side of the block, a Fabry-Perot étalon was formed, and the length of this étalon (approximately 3·8 cm.) was measured at various temperatures in terms of the wave-length of the cadmium red line by the usual method (Sears and Barrell 1934), using the three principal radiations given by a krypton lamp.

From the results, the following equation was derived:

$$L_\theta = L_{20}[1+19\cdot94(\theta-20)\times10^{-6}+13\cdot64(\theta-20)^2\times10^{-8}].$$

(b) *Amplitude corrections*

If T is the observed mean period of a damped pendulum oscillating with small amplitude, then the period T_0 for an infinitely small arc is given by

$$T_0 = T(1-\alpha_1\alpha_2/16),$$

where α_1 and α_2 are the initial and final amplitudes. This simple expression only holds when the amplitude is small, and all the definitive measurements of period were made with initial amplitudes not exceeding 0·004 radian, so that the amplitude correction never exceeded one part in a million. Since this expression for the correction is merely a theoretical one derived from the equation of motion of a damped simple harmonic motion

$$\ddot{\theta}+k\dot{\theta}+n^2\theta = 0,$$

it was necessary to determine if the amplitudes were sufficiently small to justify the application of this formula, and further, to determine if, in the actual pendulum oscillating about a material knife-edge, the period (when reduced to zero amplitude) was independent of the amplitude.

Experiments were therefore made in which the period was determined when the amplitude varied from 0·009 radian (the largest amplitude possible with the clearance available inside the vacuum enclosure) down to 0·0003 radian (the minimum amplitude which permitted satisfactory operation of the contact device). The results are given in Table V, where the periods, reduced to zero amplitude by the above formula, are shown for various values of the mean amplitude. The number of oscillations n was increased when the mean amplitude was reduced, in order that the accuracy of determination of the period should be approximately the same in each experiment. This was necessary because, with very small amplitudes the effect of the contact device on the period of the pendulum was larger than with large amplitudes, and hence the uncertainty in the initial and final fiducial signals in the period determinations was greater with small amplitudes than with larger amplitudes (see § 6, p. 86). Similar experiments were carried out with the pendulum inverted, and the results are given in Table VI. These results show that, within the range of amplitudes observed, the period, reduced to zero amplitude, remains constant to within about half a part in a million.

Table V. Effect of amplitude on period T_1 (pendulum upright)

Mean amplitude (radians)	Amplitude correction (unit 10^{-7} sec.)	Number of oscillations n	Period T_1 at 20° C	Residuals (unit 10^{-7} sec.)
0·00962	57·8	2089	1·002 8951	− 2
597	22·2	4566	8954	+ 1
311	6·0	6406	8958	+ 5
108	0·7	18249	8944	− 9
064	$0{\cdot}2_4$	24853	8966	+13
039	$0{\cdot}0_4$	62083	8945	− 8
033	$0{\cdot}0_3$	63678	8954	+ 1

T_1 (mean) = 1·002 8953; mean residual $\pm 6 \times 10^{-7}$.

Table VI. Effect of amplitude on period T_2 (pendulum inverted)

Mean amplitude (radians)	Amplitude correction (unit 10^{-7} sec.)	Number of oscillations n	Period T_2 at 20° C	Residuals (unit 10^{-7} sec.)
0·00830	42·8	4342	1·002 9012	− 1
659	26·5	6381	9019	+ 6
319	6·1	7761	9019	+ 6
319	5·4	18273	9018	+ 5
086	$0{\cdot}5_7$	62012	9017	+ 4
072	$0{\cdot}2_4$	23617	8999	−14
100	$0{\cdot}0_7$	61986	9013	0
058	$0{\cdot}0_1$	63956	9010	− 3

T_2 (mean) = 1·002 9013; mean residual $\pm 5 \times 10^{-7}$.

Hence there is no evidence of any anomalous results due to finite amplitude of oscillation on the material knife-edge, and the definitive measurements of period, all made with amplitude corrections not exceeding one part in a million, and on so-called "sharp" knife-edges, are not likely to be in error owing to any obscure cause associated with finite amplitudes of oscillation on a knife-edge of small radius. This is contrary to the recorded experience of some previous observers. Le Rolland (1922, p. 264), for example, reports that in his experiments with pendulums oscillating on knife-edges and on cylinders of various materials, he obtained anomalous results, particularly with cylinders of fairly large radii rolling on planes of relatively soft material. The apparent absence of such anomalies in the present experiments is probably due to the use of very hard bearing surfaces against hard steel knife-edges of very small radius.

In this connexion it may be remarked that, on the completion of the experiments, the chromium plated bearing surfaces were examined and on both planes a very narrow burnished line was seen where the knife-edges had been in contact. The depth of this burnished line must have been extremely small, since it was shallower than the lapping marks on the chromium plating. It probably represented a narrow area of local work-hardening due to the rolling on the knife-edge.

(*c*) *Effect of residual air pressure on period of pendulum*

The medium in which the pendulum is swinging may affect the period of vibration in three different ways:

(*a*) The weight of the pendulum is decreased by an amount equal to the weight of residual air displaced by the pendulum.

(*b*) Some of the residual air may move with the pendulum and so virtually increase its mass.

(*c*) The viscous resistance of the residual air increases the period of oscillation of the pendulum.

Since the residual air pressure was small, the corrections due to the above effects should be small; they were shown experimentally to be negligible. The measurements of the length of the pendulum were, however, carried out at atmospheric pressure, whilst the measurements of period were made when the pendulum was subjected to a much reduced pressure. A correction was therefore applied for the effect of pressure on the dimensions of the pendulum.

The total effect of the residual air on the period of the pendulum was investigated by determining the periods T_1 and T_2 at various pressures up to 144 mm. of mercury. The results are given in Tables VII and VIII, where they are compared with the calculated values determined in the case of the upright pendulum by the linear formula

$$T_1(p) = 1{\cdot}0028953 + 4{\cdot}71p \times 10^{-7},$$

and in the case of the inverted pendulum by the quadratic formula

$$T_2(p) = 1{\cdot}0029013 + (10{\cdot}3p - 0{\cdot}018p^2) \times 10^{-7}.$$

In each case the pressure p is in mm. of mercury. Each entry in Tables VII and VIII, except the first in each case, is the result of a single observation of period. From the two formulae, the values of T^2, the square of the computed period, and of g were calculated for various pressures given in Table IX. The resulting values of g show an apparent decrease with increased pressure, but so long as the pressure does not exceed 1 mm. of mercury the presence of the residual air does not affect the value of g by more than half a part in a million. In every subsequent experiment the pressure was kept below 10μ of mercury, and in all the definitive measurements of period the pressure was kept well below 5μ of mercury.

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Table VII. Effect of residual pressure on T_1 (pendulum upright)

Pressure in mm.	Observed T_1 (sec.)	Calculated T_1 (sec.)	O.–C. (unit 10^{-7} sec.)
<0·01	1·002 8953	1·002 8953	0
25·5	9070	9073	−3
38·5	9148	9134	+14
77	9317	9316	+1
111	9461	9476	−15
144	9631	9631	0
			Mean ±6

Table VIII. Effect of residual pressure on T_2 (pendulum inverted)

Pressure in mm.	Observed T_2 (sec.)	Calculated T_2 (sec.)	O.–C. (unit 10^{-7} sec.)
<0·01	1·002 9013	1·002 9013	0
0·095	9019	9014	+5
0·215	9015	9016	−1
18	9209	9192	+17
29	9309	9297	+12
57	9529	9542	−13
101·5	9865	9873	−8
			Mean ±8

Table IX. Effect of residual pressure on apparent value of g

Pressure in mm.	T_1 (sec.)	T_2 (sec.)	T^2 (sec.^{2})	g (cm./sec.^{2})
0	1·002 8953	1·002 9013	1·005 7925	981·1798
1	8958	9023	7929	1794
2	8962	9034	7930	1793
5	8977	9064	7943	1781
10	9000	9114	7959	1765
20	9047	9212	7997	1728
50	9189	9483	8140	1589
100	9424	9863	8456	1280

Finally we may consider the effect of the uniform compressive stress of 14·7 lb./sq. in. due to atmospheric pressure acting on the pendulum. Young's modulus, E, is related to the modulus of compression, K, by the formula

$$E = 3K(1-2\sigma),$$

where σ is Poisson's ratio. The value of Young's modulus for samples of the material of the pendulum rod was measured, and found to be $10{\cdot}5\times10^6$ lb./sq. in., and Poisson's ratio was found to be 0·30. Thus $E = 1{\cdot}2K$; hence the axial compression due to a compressive stress of 14·7 lb./sq. in. is $14{\cdot}7\times0{\cdot}4/10{\cdot}5$, or 0·56 part in a million. Thus the pendulum rod is about 0·6 part in a million longer *in vacuo* than in air, and so the value of g calculated from the measured value of (h_1+h_2) must be increased by 0·6 part in a million.* It should, however, be noted that the value of E used in calculating this

* This effect of compression is ignored in Appendix III, where the elastic and rigid pendulums are considered to be both subject to the same uniform compressive stress.

correction was determined by observations made with stresses of the order of 1000 lb./sq. in. (90–2150 lb./sq. in.). There is no reason to believe that the value of E associated with the small stress of 14·7 lb./sq. in. would differ appreciably from the value actually used.*

8. Measurement of length of pendulum rod

The length of the pendulum rod $L = (h_1 + h_2)$ was measured by Mr H. Barrell using the wave-length comparator of Sears and Barrell (1932). The rod was made closely 1 metre in length and the ends were carefully lapped until a proof plane showed that they were flat to within about 0·00001 in. over the majority of their area, and a level comparator showed that they were parallel to within about 0·00001 radian.

Two plates of case-hardened steel each nominally $\frac{3}{8}$ in. thick, and of the same area, $3\frac{3}{4} \times 1\frac{3}{4}$ in., as the end of the rod, were then prepared and their $3\frac{3}{4} \times 1\frac{3}{4}$ in. surfaces lapped flat and parallel. Their thicknesses were measured mechanically by comparison with standard slip gauges, so that the mean thickness of each at 20° C was known to within 0·000005 in. They were then attached, one at each end, to the rod, in the same way as were subsequently attached the blocks forming the pendulum bobs. The composite rod thus formed was then placed in the wave-length comparator, together with a standard end gauge nominally 1 m. in length, between two half-silvered glass plates, and the difference in length between the composite rod and the end gauge was determined in terms of the wave-length of the cadmium red line by the procedure described by Sears and Barrell (1932) in their determination of the relationship between wave-lengths of light and the fundamental standards of length. Thus the length of the composite rod was determined in terms of the standard end gauge, the known coefficient of expansion being used to correct the length of the light alloy rod to 20° C and a nominal coefficient of expansion for steel ($11 \times 10^{-6}/1°$ C) being assumed for the small thickness of steel involved.

These measurements were first made in January 1935 during the early stages of the work, repeated in March 1937, during the course of the definitive measurements of period, and again in April 1938, after all the definitive measurements had been completed. The results were as follows:

Length of pendulum rod at 20° C in January 1935:

$$99{\cdot}98997 \pm 0{\cdot}00002_5 \text{ cm.}$$

Length at 20° C in March 1937:

$$99{\cdot}98993 \pm 0{\cdot}00002_5 \text{ cm.}$$

Length at 20° C in April 1938:

$$99{\cdot}99000 \pm 0{\cdot}00002_5 \text{ cm.}$$

* The frequency of longitudinal vibrations of a rod under very small stresses is correctly given by the value of Young's modulus derived from much larger stresses.

The definitive measurements of period were made between July 1936 and March 1937, and between November 1937 and February 1938. The mean of the above three measurements, $99{\cdot}98997 \pm 0{\cdot}00002_5$ cm., was therefore taken to be the axial length of the pendulum rod at 20° C, measured horizontally in air at amospheric pressure.

9. Corrections to measured length of pendulum rod

The measured length of the pendulum rod at 20° C is subject to the following corrections:

(*a*) Effect of reduced pressure on length. When the rod is *in vacuo* its length is $0{\cdot}6\mu$ greater than in air, as shown in § 7; hence the correction to L is $+0{\cdot}6\mu$.

(*b*) When swinging on the slightly elastic support, the pendulum behaves as if its length were increased by $1{\cdot}5\mu$, as shown in § 4; hence the correction to L is $+1{\cdot}5\mu$.

(*c*) The compression at the knife-edge results in an apparent lengthening of the pendulum rod amounting to about $0{\cdot}5\mu$, as shown in Appendix II.

(*d*) Effect of elasticity of pendulum rod. This is dealt with in Appendix III, where the correction to L due to the elasticity of the pendulum rod is shown to be $-0{\cdot}7\mu$.

The net result of these four corrections to L is $+1{\cdot}9\mu$. The measured length of the pendulum rod at 20° C (see § 8) was taken to be 99·98997 cm.; hence the corrected value of L to be used in the formula $gT^2 = \pi^2 L$ for the computation of g is 99·99016 cm. $\pm 0{\cdot}00002_5$ cm.

10. Measurement of h_1 and h_2

The distances h_1 and h_2 of the centre of gravity of the pendulum from the ends of the pendulum rod were determined by balancing the pendulum horizontally on a length of steel wire 3·6 mm. in diameter, and measuring the distance between the line of contact of the wire and each end of the pendulum rod in turn. The measurements were made from each end of the line of contact (to eliminate effect of possible out-of-squareness of the wire with the axis of the pendulum), first with the web of the I-section vertical, and then repeated with the web horizontal. The measurements were made with sufficient accuracy with a metre steel scale, and the mean value of h_1 was found to be $26{\cdot}05 \pm 0{\cdot}02$ cm. A similar series of observations made with the pendulum balanced on a steel wire of diameter 1 mm. gave the same mean value, 26·05 cm., but with a mean residual $\pm 0{\cdot}03$ cm. Hence, since $h_1 + h_2 = 99{\cdot}99$ cm. (§ 8), $h_2 = 73{\cdot}94 \pm 0{\cdot}03$ cm., and the ratio $\frac{1}{2}\frac{h_1+h_2}{h_1-h_2} = 1{\cdot}044$, with a possible uncertainty of $\pm 0{\cdot}0013$.

The factor 1·044 was used throughout as the coefficient of $T_1^2 - T_2^2$ in the expression for T^2, and since, in all the definitive measurements of period, $T_1 - T_2$ was numerically less than 0·00001 sec., the uncertainty in the computed T^2 due to uncertainty in the values of h_1 and h_2 was always entirely negligible.

11. Summary of measurements of period

Table X gives a typical summary of five values of T_1 and five values of T_2 from which a single value of the computed period T is determined. The mean residuals ΔT_1 and ΔT_2 are given, from which the corresponding residual Δg may be calculated by means of the formula $\Delta g = 2\sqrt{\{(\Delta T_1)^2 + (\Delta T_2)^2\}}$.

Table X. Typical set of observations for determination of T_1 and T_2

Series No.	Timing interval τ sec.	Number of swings n	Period τ/n sec.	Corrections (unit 10^{-7} sec.) Temp.	Ampli-tude	Clock rate	Corrected period (sec.)	Mean period (sec.)	Residuals (unit 10^{-7} sec.)
Pendulum upright:									
273	11364·7456	11332	1·002 8897	+60	−6	+11	1·002 8962		−1
274	11789·9883	11756	8911	+48	−3	+11	8967		+4
275	13859·9428	13820	8902	+57	−4	+11	8966	T_1	+3
276	11354·7563	11322	8932	+29	−7	+11	8965	1·002 8963	+2
277	11465·0704	11432	8928	+21	−4	+11	8956		−7
								Mean	±3
Pendulum inverted:									
278	11679·8195	11646	1·002 9040	0	−7	+12	1·002 9045		−5
279	16250·0425	16203	9033	+15	−7	+12	9053		+3
280	11900·4887	11866	9065	−20	−7	+12	9050	T_2	0
281	15494·8549	15450	9032	+10	−5	+12	9049	1·002 9050	−1
282	15385·5375	15341	9032	+15	−6	+12	9053		+3
								Mean	±2

Altogether eighteen definitive values of T^2 were obtained, each depending on ten measurements of period similar to the typical set summarized in Table X. They are summarized in Table XI in which set nine represents the results shown in Table X.

Table XI. Computed periods

1 Set no.	2 Date	3 Radius and position of knife-edge	4 T_1 sec.	5 Mean residuals $\Delta T_1 \times 10^7$ sec.	6 T_2 sec.	7 Mean residuals $\Delta T_2 \times 10^7$ sec.	8 T^2 (sec.^{2})
1	July 1936	13μ A	1·002 8875	±5	1·002 8784	±3	1·005 7933
2	to	13 B	8855	4	8800	4	7853
3	Oct. 1936	3·5 A	8908	4	8866	8	7945
4		3·5 B	8895	5	8886	6	7883
5		17 A	8851	3	8744	8	7902
6		17 B	8855	6	8756	6	7901
7	Nov. 1936	2·5μ A	1·002 8969	±5	1·002 9072	±5	1·005 7909
8	to	2·5 B	8961	4	9068	3	7889
9	Mar. 1937	3 A	8963	3	9050	2	7915
10		3 B	8954	6	9042	5	7896
11		3 A	8958	4	9057	4	7892
12		3 B	8961	3	9052	7	7907
13	Nov. 1937	1·5μ A	1·002 8928	±7	1·002 8923	±8	1·005 7945
14	to	1·5 B	8909	9	8922	5	7888
15	Feb. 1938	2·5 A	8927	5	8965	11	7896
16		2·5 B	8929	4	8959	8	7909
17		2·5 A	8917	2	8937	6	7896
18		2·5 B	8927	4	8945	3	7918

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With each knife-edge one value of the computed period T (depending on five observations of T_1 and five of T_2) was determined, and then the knife-edge was reversed end for end on its support. A further five observations of T_1 and five of T_2 then resulted in another value of T, which usually differed slightly from the first value. The two possible positions of the knife-edge are designated by A and B in column 3 of Table XI. The pendulum remains nominally unchanged during such a set of twenty observations of period, but it should be remembered that each time the pendulum is reversed end for end (to determine T_1 and T_2) the knife-edge has to be removed and replaced. The above table shows evidence that reversing the knife-edge end for end may slightly affect the period; hence small variations may also occur when the knife-edge is replaced after reversing the pendulum.

12. Summary of values of g

The corrected length of the pendulum (h_1+h_2) at 20° C is 99·99016 cm., as shown in § 9, and from the squares of the computed periods T^2 given in column 8 of Table XI for various values of the knife-edge radius, the values of g given in column 5 of Table XII are calculated by means of the formula $gT^2 = \pi^2 (h_1+h_2)$.

Table XII. Final values of g

1 Series	2 Date	3 Radius in μ and position of knife-edge	4 T^2 sec.^{2}	5 g cm./sec.^{2}	6 Correction for radius of knife-edge mgal.	7 Corrected g cm./sec.^{2}	8 Residuals Δ mgal.
1	July 1936	13 A	1·005 7933	981·1791	−0·8	981·1783	−3·2
	to	B	7853	1869	−0·8	1861	+4·6
2	Oct. 1936	3·5 A	7945	1779	−0·2	1777	−3·8
		B	7883	1839	−0·2	1837	+2·2
3		17 A	7902	1821	−1·1	1810	−0·5
		B	7901	1822	−1·1	1811	−0·4
4	Nov. 1936	2·5 A	1·005 7909	981·1814	−0·2	981·1812	−0·3
	to	B	7889	1833	−0·2	1831	+1·6
5	Mar. 1937	3 A	7915	1808	−0·2	1806	−0·9
		B	7896	1827	−0·2	1825	+1·0
6		3 A	7892	1831	−0·2	1829	+1·4
		B	7907	1816	−0·2	1814	+0·1
7	Nov. 1937	1·5 A	1·005 7945	981·1779	−0·1	981·1778	−3·7
	to	B	7888	1834	−0·1	1833	+1·8
8	Feb. 1938	2·5 A	7896	1827	−0·2	1825	+1·0
		B	7909	1814	−0·2	1812	−0·3
9		2·5 A	7896	1827	−0·2	1825	+1·0
		B	7918	1805	−0·2	1803	−1·2

The values of g thus obtained are then corrected for the effect of the small radius of curvature of the knife-edge, in accordance with the formula of § 3. The values of these corrections are given in column 6, and the final values of g are tabulated in column 7.

The mean value of g is 981·1815 cm./sec./sec. and column 8 gives the residuals, the mean residual being $\pm 1{\cdot}6$ mgal.

The final value of g, **981·1815 cm./sec./sec.**

refers specifically to the position occupied by the pendulum apparatus in Room 11 of the Metrology Department of The National Physical Laboratory, Teddington:

Latitude	51° 25′ 14″ N.
Longitude	0° 20′ 21″ W. approximately.
Height above M.S.L. ...	10 m.

This value of g, when related to Potsdam by relative determinations (Bullard and Jolly 1936),* gives

$$g\ (\text{Potsdam}) = 981{\cdot}261_2,$$

whilst the absolute determination of Kühnen and Furtwängler gave the value

$$g\ (\text{Potsdam}) = 981{\cdot}274.$$

On the other hand, the recent absolute determination at the Bureau of Standards, Washington, by Heyl and Cook (1936), when related to Potsdam by relative determinations, gave

$$g\ (\text{Potsdam}) = 981{\cdot}254.$$

Hence there is some evidence that the value of g obtained by Kühnen and Furtwängler for Potsdam is too large by between 10 and 20 parts in a million.

13. Accuracy of final value of g

In discussing the probable accuracy of the final value of g, we may first consider the variations in period shown in a typical set of five values of T_1 and five of T_2, as in Table X. The mean of all the values of ΔT_1 is $\pm 4{\cdot}6\times 10^{-7}$ and of ΔT_2 $\pm 5{\cdot}6\times 10^{-7}$ (Table XI). It is evident that, so long as the pendulum remains undisturbed on a given knife-edge, its period is constant, and can be measured to within about ± 5 parts in 10^7. This corresponds with an accuracy of about ± 14 parts in 10^7 in the value of g.

The estimated uncertainties in the corrections applied to the measured values of T_1 and T_2 are summarized in Table XIII, together with the uncertainty in the correction to g due to the geometrical effect of the finite radius of curvature of the knife-edge.

Finally, the corrections to the measured length of the pendulum should be considered. They are summarized as follows:

Nature of correction	Value of correction	Estimated uncertainty	Resultant uncertainty in mean value of g mgal.
(*a*) Elasticity of pendulum support	$+1{\cdot}5\mu$	$\pm 0{\cdot}05\mu$	$\pm 0{\cdot}05$
(*b*) Elasticity of the pendulum itself	$-0{\cdot}7\mu$	$\pm 0{\cdot}1\mu$	$\pm 0{\cdot}1$
(*c*) Difference in length in air and *in vacuo*	$+0{\cdot}6\mu$	$<0{\cdot}05\mu$	$<0{\cdot}05$
(*d*) Compression at knife-edge	$+0{\cdot}5\mu$	$\pm 0{\cdot}25\mu$	$\pm 0{\cdot}25$

* There is an error in Table V of the paper by Bullard and Jolly, p. 450. The value of g at Teddington, in terms of the Potsdam value, should appear as 981·1953 instead of 981·1817.

The probable error of measurement of the length (h_1+h_2) is $\pm 0{\cdot}25\mu$ (§ 8); hence the corrected value of (h_1+h_2) used in the computation of g may be in error by about $\pm 0{\cdot}4\mu$, corresponding with an error in g of about 0·4 mgal.

The combined effect on g of a possible error of between $\pm 1{\cdot}3$ and $\pm 1{\cdot}4$ mgal. due to uncertainties in period measurements and $\pm 0{\cdot}4$ mgal. due to uncertainties in length measurements is therefore about $\pm 1{\cdot}4$ mgal.

Table XIII. Summary of estimated probable errors

Nature of correction	Estimated uncertainty of correction (parts in 10^7)	Resultant uncertainty in value of g mgal.
Corrections to T_1 and T_2:		
(a) Temperature	± 2	$\pm 0{\cdot}6$
(b) Amplitude	± 1	$\pm 0{\cdot}3$
(c) Clock rate	± 1	$\pm 0{\cdot}3$
(d) Uncertainty in interpretation of chronograph records	± 4	$\pm 1{\cdot}1$
Total uncertainty in correction to T_1 and T_2	$\pm 4{\cdot}7$	$\pm 1{\cdot}3$
Corrections to g for effect of radius of knife-edge	± 1	$\pm 0{\cdot}1$
Corresponding total uncertainty in value of g	—	$\pm 1{\cdot}3$ to $\pm 1{\cdot}4$

This corresponds closely with the observed variations in the experimental results. The values of g in Table XII range from 981·1777 to 981·1861, with a mean residual of $\pm 1{\cdot}6$ mgal., the Gaussian probable error of an individual entry for g in Table XII being $\pm 1{\cdot}4$ mgal. Assuming there were no systematic errors, the probable error of the mean value of g was calculated by means of the Gaussian formula p.e. $= \pm 0{\cdot}675\sqrt{\{\Sigma\Delta^2/n(n-1)\}}$, using the values of the residuals Δ given in the last column of Table XII, and found to be $\pm 0{\cdot}34$ mgal. Although the mean of eighteen determinations, 981·1815 cm./sec.^{2}, has a probable error of only $\pm 0{\cdot}34$ mgal., the mean residuals $\pm 1{\cdot}6$ mgal. (observed) and $\pm 1{\cdot}4$ mgal. (estimated) indicate that the most likely value of g is $981{\cdot}181_5$ cm./sec.^{2} with a possible range of about $\pm 1{\cdot}5$ mgal.

14. Conditions requisite for accurate results

It is generally recognized that the anomalies caused by the necessity of swinging a pendulum about a material edge, rather than about a geometrical point or line, may give rise to appreciable errors in the absolute determination of g. The material, as well as the geometrical form, of the knife-edge, and also the nature of the bearing surface which the knife-edge supports all have a marked effect on the period of a pendulum.

Le Rolland (1922, p. 219) has described anomalous results obtained with pendulums oscillating on knife-edges and on cylinders of various materials, and taking into account

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his results, together with general consideration of other more recent experience with knife-edges, it was thought that reliable results would be obtained with a hardened steel knife-edge having a perfectly straight edge formed by the intersection of two optically flat planes, the pendulum being provided with bearing surfaces of chromium plating, ground and lapped optically flat. The experiments of § 3 showed that knife-edges of large radius of curvature gave anomalous values of g, but the anomalies did not appear to become appreciable until the radius was of the order of five times the largest radius of the "sharp" knife-edges actually used in the definitive measurements. The combination of sharp steel knife-edge against chromium plated plane did not exhibit any of the other anomalies observed by Le Rolland with certain combinations of knife-edges and planes. Thus, the results of section 7 (b) showed that with a hardened steel knife-edge of very small radius, the period, when reduced to the limiting value for an infinitely small arc, was independent of the amplitude, and results obtained by successive reversals of the pendulum end for end on the same knife-edge gave no indication of progressive wear of the knife-edge, or of any anomalous progressive changes in period. The results of such a series of reversals made in the early stages of the work are shown in Table XIV.

Table XIV. Successive reversals on the same (steel) knife-edge

Date	T_1 at 20° C sec.	Mean T_1 sec.	ΔT_1 (unit 10^{-7} sec.)	T_2 at 20° C sec.	Mean T_2 sec.	ΔT_2 (unit 10^{-7} sec.)	T^2 and g (apparent value of g, not corrected for radius of knife-edge)
26–28. x. 35	1·002 8917		−24				
	8955		+14				
	8947	1·002 8941	+ 6				
	8942		+ 1				
	8946		+ 5				
		Mean	±10				
30–31. x. 35				1·002 9008		+ 9	$T^2=1{\cdot}005\,7902$
				8976		−23	$g=981{\cdot}1821$
				8990	1·002 8999	− 9	
				9018		+19	
				9005		+ 6	
					Mean	±13	
2–5. xi. 35	1·002 8971		+34				
	8914		−23				
	8930	1·002 8937	− 7				
	8918		−19				
	8952		+15				
		Mean	±20				
8–12. xi. 35				1·002 8995		0	$T^2=1{\cdot}005\,7894$
				8981		−14	$g=981{\cdot}1829$
				8980	1·002 8995	−15	
				9020		+25	
				9001		+ 6	
					Mean	±12	

In these (early) experiments, the number of oscillations observed was about 5000; hence the mean residuals of T_1 and T_2 are larger than those obtained in the definitive experiments where n exceeded 10,000.

It might have been thought that a "sharp" knife-edge having a radius of only 3μ would break down under the weight of the pendulum, but the length of bearing surface ($1\frac{3}{4}$ in.) in contact with the knife-edge is considerable, and both the bearing surface and edge are sufficiently straight to distribute the load uniformly over at least $1\frac{1}{2}$ in. of the knife-edge. Actually, a radius of the order of 2μ was achieved with several of the knife-edges, but in some cases the radius of curvature when remeasured after the knife-edge had been used under the pendulum was found to be greater than this. During the course of the definitive experiments with the steel knife-edges, the results of the determination of the absolute value of g at Washington by Heyl and Cook (1936) became available, and their discussion of the merits of various materials for knife-edges and planes confirmed the view that the combination hard steel against chromium plating should give satisfactory results, and it was therefore considered unnecessary to experiment with any other forms of knife-edge. Actually, a few experiments were made with a knife-edge of gun-metal in order to obtain some idea of the order of magnitude of the anomalies likely to be introduced by a knife-edge of material very appreciably softer than steel. The knife-edge was of the same form as the steel ones, and made by the same technique, although it is obvious that the same perfection of lapping could not be expected with such a soft material. Its measured radius of curvature was initially 22μ, and the results obtained with it are given in Table XV.

Table XV. Period of pendulum on gun-metal knife-edge

Radius of curvature of knife-edge, initially, 22μ

Date	Number of oscillations observed, n	Corrections (unit 10^{-7} sec.) Temperature	Amplitude	Period, T_1 sec.
21. iv. 37	11381	−48	−7	1·002 8447
22. iv. 37	11182	+30	−3	8391
22. iv. 37	11122	+ 4	−6	8381
23. iv. 37	10684	+14	−5	8361
23. iv. 37	12503	−24	−7	8352
26. iv. 37	21270	+41	−2	8301
27. iv. 37	11029	+134	−4	8296
27. iv. 37	10466	+95	−5	8280

Radius of knife-edge at conclusion of experiments, 33μ.

It is seen that, not only is the period appreciably smaller than with a steel knife-edge of similar radius, but the value of the period progressively decreased with each determination; the total decrease was about 17 parts in a million during the six days occupied by the eight determinations. Before reversing the pendulum, the radius of curvature of the knife-edge was remeasured and was found to have increased to 33μ. A similar series of observations of the period T_2 was now made with the pendulum reversed, and the period T_2 decreased progressively from 1·002 755 to 1·002 678 sec., whilst the radius of curvature of the knife-edge still further increased from 33 to 60μ. No suggestion of this effect was observed in any of the determinations using steel knife-edges, and it was concluded that hardened steel, particularly because the technique of lapping accurately flat surfaces and straight edges in steel is now well established, is a suitable material for making knife-edges for gravity pendulums. Heyl and Cook report (1936) that their best results were obtained with chrome steel knife-edges in conjunction with planes of stellite or of fused silica.

Two other requirements for accurate results tend unfortunately to be mutually inconsistent. In order to minimize compression between the knife-edge and bearing plane it is desirable to use a light pendulum, but in order to reduce the elasticity of the pendulum and to increase the length of knife-edge in contact with the plane bearing surface, a fairly massive form of pendulum must be used. In the present pendulum the weight has been reduced without sacrificing rigidity by using an I-section of light alloy. Kühnen and Furtwängler in the Potsdam determination used pendulums appreciably lighter than that used in the present determination, but their estimated elasticity corrections were 4 or 5 times as large as in the present work. There is no evidence in the present work that the pendulum was too heavy for the steel knife-edges, even though the pressure on the knife-edge, estimated from the data given in Appendix II, amounted to about 100 tons/sq. in. for a radius of 25μ, and to as much as 200 tons/sq. in. in the case of the "sharp" knife-edges of radius about 5μ.* Whenever there was a measurable change in radius of curvature of the knife-edge before and after use, the evidence (constancy of period, and absence of visible damage) indicated that the change occurred the first time the pendulum was swung on the knife-edge, and in no case was the radius increased to anything approaching the value necessary to produce an appreciable change in the apparent value of g.

15. Acknowledgements

In conclusion, the author wishes to thank many of his colleagues at The National Physical Laboratory for their help in connexion with this work. In particular, the author is very grateful to Mr Sears, Superintendent of the Metrology Department, for his interest and advice during the course of the work and in the preparation of this paper; to Mr Turner and the staff of the drawing office, who worked out many of the details of the design of the apparatus, and produced the drawings used to illustrate this paper; to Mr Knoyle of the Metrology workshop, who undertook the precision work of grinding and lapping the ends of the pendulum rod flat and parallel, and who made all the steel knife-edges—a technical achievement of high order; to Mr Barrell, who measured the length of the pendulum rod; to Mr Turl of the Metrology Instrument Workshop, who made the three platinum resistance thermometers; to the Physics

* These estimated pressures are so large that some breakdown of the knife-edge might be expected to occur; actually there was no appreciable visible deterioration of the edge. Possibly the theoretically derived estimate of the area of contact is too small, so that the estimated pressure is larger than the real pressure. On the other hand, the calculated pressures are appreciably lower than the generally accepted value of the "flow stress", about 300 tons/sq. in.

Department, who calibrated the thermometers; to the Engineering Department, who determined the elastic constants of the material of the pendulum rod; and to the Line-Standard Section of the Metrology Department, who determined the linear coefficient of expansion of the material of the pendulum rod.

Appendix I. Theory of reversible pendulum swinging on fixed knife-edge of finite radius of curvature

In fig. 10 the circle with centre C and radius r represents a cross-section of the cylindrical knife-edge and the straight line P represents the flat surface of the pendulum just in contact with the knife-edge. As the weight of the pendulum is taken by the knife-edge, the pendulum approaches the position of equilibrium P_1, a total compression of amount α occurring at the contact. The form of the boundary surface between the pendulum and knife-edge is now a portion of the circular arc XY of radius of the order $2r$. The radius of curvature of the boundary surface would be very closely $2r$ if the materials of the knife-edge and plane were identical.

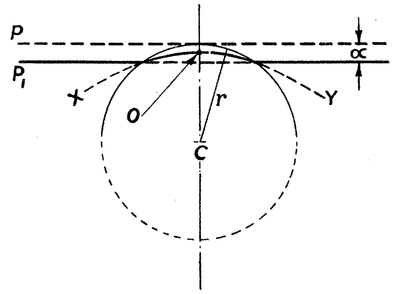

Fig. 10

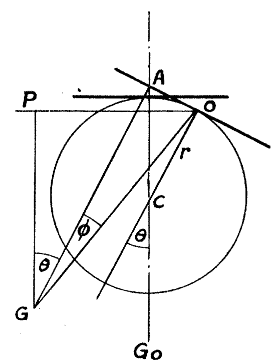

Fig. 11

Figs. 10 and 11. Theory of pendulum on knife-edge of finite radius.

In fig. 11 a circle with centre C and radius r is drawn to represent a section of the knife-edge about which the plane P in the pendulum is assumed to roll without slipping. In the position of rest (pendulum vertical) the centre of gravity of the pendulum is at G_0, and OAG represents the pendulum when displaced through an angle θ. The centre of gravity is then at G, and the instantaneous centre of rotation is at O.

$AG = h_1$ is the distance of the centre of gravity from the plane P, which is distant α_1 above the actual plane on the pendulum.

Now
$$\frac{1}{g}\frac{d^2\theta}{dt^2} = OP/(k^2+GO^2),$$

where k is the radius of gyration of the pendulum about its centre of gravity, G.

But

$$OP = OG\sin(\theta+\phi) = (h_1^2+r^2\theta^2)^{\frac{1}{2}}\sin(\theta+r\theta/h_1) = h_1\sin(h_1+r)\,\theta/h_1 = \theta(h_1+r)$$

very closely, ϕ being the very small angle AGO.

Then $$\frac{1}{g}\frac{d^2\theta}{dt^2} = (h_1+r)\,\theta/(k^2+h_1^2),$$

and the period T_1* is therefore given by

$$gT_1^2/\pi^2 = (k^2+h_1^2)/(h_1+r). \quad (1)$$

The pendulum is now reversed, and h_2 is the distance from the new virtual bearing surface to the centre of gravity. Then, as before,

$$gT_2^2/\pi^2 = (k^2+h_2^2)/(h_2+r). \quad (2)$$

Subtracting (2) from (1), we have, to a first approximation,

$$\begin{aligned} g(T_1^2-T_2^2)/\pi^2(h_1-h_2) &= 1-k^2/h_1h_2+rk^2(h_1+h_2)/h_1^2h_2^2 \\ &= 1-k^2(1-r/h_1-r/h_2)/h_1h_2. \end{aligned}$$

We may note here that since r is very small compared with both h_1 and h_2,

$$g(T_1^2-T_2^2)/\pi^2(h_1-h_2) = 1-k^2/h_1h_2 \text{ approximately,}$$

so that when T_1 and T_2 are nearly equal, k^2/h_1h_2 is very nearly equal to 1.

Now adding (1) and (2),

$$g(T_1^2+T_2^2)/\pi^2(h_1+h_2) = \{k^2+h_1h_2+(2k^2+h_1^2+h_2^2)\,r/(h_1+h_2)\}/(h_1+r)\,(h_2+r). \quad (3)$$

Similarly by subtracting (2) from (1),

$$g(T_1^2-T_2^2)/\pi^2(h_1-h_2) = \{h_1h_2-k^2+r(h_1+h_2)\}/(h_1+r)\,(h_2+r). \quad (4)$$

Now, adding (3) and (4), we obtain

$$\begin{aligned} g/\pi^2[(T_1^2+T_2^2)/(h_1+h_2)+(T_1^2-T_2^2)/h_1-h_2)] \\ = 2+\{2r(k^2-h_1h_2)/(h_1+h_2)-2r^2\}/(h_1+r)\,(h_2+r). \end{aligned}$$

In this expression both r and $(k^2-h_1h_2)$ are very small. In estimating the small correction term on the right we may therefore legitimately substitute the approximate value h_1h_2 for $(h_1+r)\,(h_2+r)$ in the denominator. Making this substitution, and dividing by 2, we obtain

$$\begin{aligned} g/2\pi^2[T_1^2+T_2^2)/(h_1+h_2)+(T_1^2-T_2^2)/(h_1-h_2)] \\ &= 1-r(1-k^2/h_1h_2)/(h_1+h_2)-r^2/h_1h_2 \\ &= 1-r[(1-k^2/h_1h_2)+r/h_1+r/h_2]/(h_1+h_2). \quad (5) \end{aligned}$$

* T_1 is strictly the half-period; but since a pendulum approximately 1 m. in length is usually referred to as a "seconds pendulum", the half-period has been described throughout as the "period of the reversible pendulum".

It is shown below that the correction term on the right-hand side of equation (5) was entirely negligible under the conditions of the present determinations, and we may therefore write

$$g/2\pi^2[(T_1^2+T_2^2)/(h_1+h_2)+(T_1^2-T_2^2)/(h_1-h_2)] = 1. \tag{6}$$

If we now introduce the "computed time", T, defined by

$$T^2 \equiv (T_1^2+T_2^2)/2+(T_1^2-T_2^2)\,(h_1+h_2)/2(h_1-h_2), \tag{7}$$

or

$$T^2 \equiv (h_1T_1^2-h_2T_2^2)/(h_1-h_2), \tag{8}$$

we have finally the equation

$$gT^2/\pi^2 = h_1+h_2, \tag{9}$$

from which the values of g are computed.

Actually, in the final experiments, T_1 and T_2 were made equal to well within 0·00002 sec.; h_1 was approximately 74 cm., and h_2 approximately 26 cm., (T_1+T_2) was closely 2 sec., and r was less than 20μ.

Hence

$$r/h_1 < 27\times 10^{-6},$$

$$r/h_2 < 77\times 10^{-6},$$

and

$$r/(h_1+h_2) < 20\times 10^{-6}.$$

Further, substituting the appropriate values in equation (5), we find

$$(981\times 4\times 10^{-5})/(9{\cdot}87\times 48) > 1-k^2/h_1h_2$$

or

$$1-k^2/h_1h_2 < 83\times 10^{-6}.$$

Hence the total correction term in equation (5) is less than $20\times(83+27+77)\times 10^{-12}$, or less than 4×10^{-9}.

Even in the case of the experimental investigation of the effect of finite radii in § 3, where the greatest value of r was 282μ, and the greatest difference between T_1 and T_2 was 0·000324 sec., the maximum value of the correction term was less than 8×10^{-7}.

Thus the purely geometrical effect of the finite radius of the knife-edge is entirely negligible, and it only remains to determine the small correction $(\alpha_1+\alpha_2)$ or, since the materials of the two bearing planes are similar, 2α, which must be added to the measured length of the pendulum in order to obtain the effective value of (h_1+h_2). This is dealt with in Appendix II.

Appendix II. Elastic compression at knife-edge

The compression between a cylindrical knife-edge and a plane can be calculated by regarding it as a special case of the compression between two cylinders with their axes perpendicular.

The general equations for the elastic compression when two bodies are pressed together in local contact are given by Love (1920). They were originally due to Hertz (1881, 1896).

If $2A$ is the sum of the two principal curvatures of one of the bodies, and $2B$ is the sum of the two principal curvatures of the other, then the general equations connecting the stress and strain in terms of the eccentricity, e, and the semi-major axis, a, of the ellipse of contact, may be written

$$Aa^3 = 3\vartheta P\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}}} \equiv -3\vartheta P\frac{1}{e}\frac{dE}{de}, \tag{1}$$

$$Ba^3 = 3\vartheta P\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} \equiv 3\vartheta P\frac{1}{e}\frac{dK}{de}, \tag{2}$$

and

$$\alpha = \frac{3\vartheta P}{\alpha}\int_0^{\pi/2}\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}}} \equiv \frac{3\vartheta P}{\alpha}K. \tag{3}$$

K and E are complete elliptic integrals of the first and second classes respectively, e being the modulus.

P is the total pressure between the cylinders; in the case under consideration, P is the weight of the pendulum.

$$\vartheta \equiv \frac{3+4N}{4\pi C(3+N)}, \tag{4}$$

where C is the modulus of rigidity of the material* forming the cylinders, and N is the ratio of modulus of rigidity to modulus of compression.

α is the total elastic compression, i.e. the distance by which the centres of the cylinders approach together when compressed.

In the actual case of the pendulum, the cylindrical knife-edge is of steel, but the bearing surface on the pendulum is a thin layer of chromium plated on delta metal. The elastic constants N and C for steel are sufficiently well known for the purpose of these calculations, but no information is available regarding the elastic constants of chromium in the form of plated films. Since however the compression α is small, it is probably sufficiently accurate to assume the elastic constants of the chromium plating to be the same as those of steel.

From equations (1) and (2), starting with four arbitrary small values of e_1, defined by $e_1^2 \equiv 1-e^2$, and making use of the subsidiary relations between the elliptic integrals K and E, viz.

$$\frac{dE}{de} = (E-K)/e \quad \text{and} \quad \frac{dK}{de} = (E-e_1^2K)/ee_1^2,$$

together with the expressions for K and E which hold when e is very nearly unity, viz.

$$K = \log\frac{4}{e_1}+\frac{e_1^2}{4}\left(\log\frac{4}{e_1}-1\right)+\dots$$

and

$$E = 1+\frac{e_1^2}{2}\left(\log\frac{4}{e_1}-\frac{1}{2}\right)+\dots,$$

* If the two bodies in contact are of different materials we may write

$$\vartheta = \tfrac{1}{2}\vartheta_1+\tfrac{1}{2}\vartheta_2,\ \text{where}\ \vartheta_1\equiv\frac{3+4N_1}{4\pi C_1(3+N_1)}\ \text{and}\ \vartheta_2\equiv\frac{3+4N_2}{4\pi C_2(3+N_2)}.$$

Table XVI has been calculated, the values of A/B being derived from dE/de and dK/de in accordance with equations (1) and (2).

Table XVI

e_1	K	$-\frac{dE}{de}\Big/e$	$\frac{dK}{de}\Big/e$	A/B
0·0001	10·597	9·597	10^8	$9{\cdot}597\times10^{-8}$
0·00005	11·290	10·290	4×10^8	$2{\cdot}573\times10^{-8}$
0·00002	12·206	11·206	25×10^8	$4{\cdot}482\times10^{-9}$
0·00001	12·899	11·899	10^{10}	$11{\cdot}899\times10^{-10}$

In order to apply these results to the case of a cylindrical knife-edge pressed against a plane, we must assume that the plane is a cylinder of very large radius R, and that the knife-edge, which has a small finite radius r, is not straight, but has a very large radius R in the plane perpendicular to the section which has the radius r.

Then half the sum of the two principal curvatures of the "plane" may be written $A = 1/2R+1/\infty = 1/2R$, and for the knife-edge, $B = 1/2r+1/2R = (R+r)/2Rr$. Hence $\frac{A}{B}=\frac{r}{R+r}=\frac{r}{R}$, since R is very large compared with r, the radius of curvature of the knife-edge, and $A/B = 2rA$, so that it is possible, for any value of the radius of curvature r of the knife-edge, to calculate A from A/B, and then, from the equations

$$a^3 = 3\vartheta P/A\left(-\frac{dE}{de}\Big/e\right) \quad \text{and} \quad \alpha = 3\vartheta PK/a,$$

to determine a and α with the aid of the above table. a is the semi-major axis of the ellipse of contact, and the value of α required is that for which $2a$ is the length of knife-edge in contact with the plane bearing surface on the pendulum.

The modulus of compression for steel may be taken to be $22{\cdot}8\times10^6$ lb./sq. in. and the modulus of rigidity $11{\cdot}4\times10^6$ lb./per sq. in.; hence $N = 0{\cdot}5$, and $\vartheta = 9{\cdot}97\times10^{-9}$. The actual weight of the reversible pendulum was 22·88 lb. and the length of knife-edge in contact with the plane was 1·75 in., so that $a = 0{\cdot}875$ in.

The radii of the knife-edges actually used in the definitive measurements of period ranged from about 0·001 in. down to about 0·0001 in. Taking the case of $r = 0{\cdot}001$ in., we have $A = 500A/B$. Then, starting from the values of A/B given in Table XVI, the various stages in the computation of a and α are given in Table XVII:

Table XVII. Computation of α for $r = 0{\cdot}001$ in.

A/B	A (in.)$^{-1}$	a (in.)	$\alpha\times10^6$ (in.)
$9{\cdot}597\times10^{-8}$	$4{\cdot}798\times10^{-5}$	0·515	14·1
$2{\cdot}573\times10^{-8}$	$1{\cdot}286\times10^{-5}$	0·818	$9{\cdot}4_5$
$4{\cdot}482\times10^{-9}$	$2{\cdot}241\times10^{-6}$	1·507	$5{\cdot}5_5$
$11{\cdot}899\times10^{-10}$	$5{\cdot}949\times10^{-7}$	2·392	3·7

By intrapolation it is estimated that the value of α corresponding with $a = 0{\cdot}875$ in. is approximately 9×10^{-6} in.

For a knife-edge of radius $r = 0{\cdot}0001$ in. we have $A = 5000\ A/B$, and the values of a and α are given in Table XVIII.

TABLE XVIII. COMPUTATION OF α FOR $r = 0{\cdot}0001$ IN.

A/B	A (in.)$^{-1}$	a (in.)	$\alpha \times 10^6$ (in.)
$9{\cdot}597 \times 10^{-8}$	$4{\cdot}798 \times 10^{-4}$	0·239	30·3
$2{\cdot}573 \times 10^{-8}$	$1{\cdot}286 \times 10^{-4}$	0·381	20·3
$4{\cdot}482 \times 10^{-9}$	$2{\cdot}241 \times 10^{-5}$	0·700	$11{\cdot}9_5$
$11{\cdot}899 \times 10^{-10}$	$5{\cdot}949 \times 10^{-6}$	1·110	$7{\cdot}9_5$

Again by intrapolation, the value of α corresponding with $a = 0{\cdot}875$ in. is approximately $9{\cdot}5 \times 10^{-6}$ in.

Hence the approximate value for the compression at the knife-edge would appear to be between 9 and $9{\cdot}5 \times 10^{-6}$ in. had both the surfaces in contact been steel, and as the correction has to be applied to both h_1 and h_2 the total correction to $(h_1 + h_2)$ would be about $+19 \times 10^{-6}$ in., giving a correction to g of about 5 parts in 10 millions. Since the modulus of rigidity of the chromium plating is almost certainly larger than that of steel, it might be assumed that the actual correction applicable to the gravity pendulum is less than half a part in a million; on the other hand, assuming that the compression is not entirely taken by the chromium plating, and that the underlying delta metal is subject to the compression due to the knife-edge, corresponding calculations have been made, using the value $\vartheta = \frac{1}{2}\vartheta_1 + \frac{1}{2}\vartheta_2 = 15{\cdot}7 \times 10^{-9}$ applicable to the case of steel against delta metal, and the values of α were found to be 14×10^{-6} in. for a radius of 0·001 in. and 15×10^{-6} in. for a radius of 0·0001 in. In this case the resulting correction to g would be about 7 parts in 10 millions. Hence on either assumption, the correction appears to be appreciably less than one part in a million, and a correction to g of half a part in a million was therefore adopted.

The above theoretical method of treatment of the problem of the compression between a plane and a cylinder, based on the equations of Hertz, owes much to some unpublished notes on the compression of cylindrical gauges written in 1921 by Mr C. H. Grant, at one time a colleague of the author.

Some experimental determinations of the compression of steel cylinders between steel planes have been described by H. Bochmann in a dissertation "Die Abplattung von Stahlkugeln und Zylindern durch den Messdruck". The experiments were made with cylinders ranging from 5·7 mm. down to 0·18 mm. in diameter, under pressures from 1 to 10 kg., and the formula

$$2\alpha = 0{\cdot}923\,\frac{P}{L}\left(\frac{1}{D}\right)^{\frac{1}{3}}$$

was derived. Here 2α is the total compression, in microns, of a cylinder of diameter D mm. between two parallel planes, L being the length of cylinder in mm. in contact with each of the planes, and P being the total pressure in kg. Extrapolating this formula for the small radii of the knife-edges we obtain $\alpha = 11{\cdot}5 \times 10^{-6}$ in. for a radius of 0·001 in.

and $\alpha = 24{\cdot}7 \times 10^{-6}$ in. for a radius of 0·0001 in. The value for a cylinder of radius 0·001 in. is not very different from that calculated above from the equations of Hertz, but the value for a radius of 0·0001 in. is about 2·6 times that obtained by calculation. Probably such a large extrapolation of Bochmann's experimental formula is unjustifiable, and all that can be said of this experimental evidence is that it tends to confirm the order of magnitude of the compression derived from purely theoretical considerations.

Appendix III. Effect of the elasticity of a pendulum on its period of oscillation

The possibility of computing the effect of the elasticity of a pendulum on its period of oscillation has been considered in connexion with previous absolute determinations of g, notably by F. R. Helmert (1898) and by E. Almansi (1898, 1899). The calculations made by the latter are complicated and not entirely free from arithmetical errors; they relate to the particular case of a pendulum having a hollow rod of circular section, of the type used by Kühnen and Furtwängler in their absolute determination of g. One of the fundamental assumptions made by Almansi was that the bending moment acting at any instant on any point of the elastic pendulum could be calculated from the forces which would act on the corresponding point of rigid pendulum when subject to the same angular displacement. This assumption is not strictly correct, but it is a sufficiently close approximation whereby the forces acting may be calculated in order to determine the configuration of the elastic pendulum at any instant. Almansi made use of the general equations of the theory of elastic strain, but this appears to be an unnecessary complication. For the purpose of calculating the elasticity correction of the present fairly rigid pendulum it was considered sufficient to use the usual approximate formulae for bending moments and elastic strain energy.

The reversible pendulum may be considered to consist essentially of an elastic rod of I-section, to each end of which is attached a solid "bob" of rectangular section. These bobs, being of much more substantial cross-section than the pendulum rod itself, are less flexible than the pendulum rod.

Let M_0 (fig. 12) be the mass of the pendulum rod, and l its length. M_1 is the mass of the upper bob, and M_2 that of the lower bob. The pendulum oscillates about a knife-edge at O, which is taken as the origin of the moving axes Ox, Oy and of the fixed axes Ox_1, Oy_1 as shown in fig. 12.

h_0 $(= l/2)$, h_1 and h_2 are the distances of the centres of gravity of the component parts, M_0, M_1 and M_2, of the pendulum measured from the origin O, and r_0, r_1 and r_2 are the corresponding radii of gyration of the component parts about O.

Consider now a rigid pendulum having the same geometrical form as the elastic pendulum when in an unstrained state. If E_e, E_p and E_k are the elastic, potential and kinetic energy respectively of the rigid pendulum and E'_e, E'_p and E'_k the corresponding terms for the elastic pendulum at any instant, then we may write

$$\delta E_e = E'_e - E_e, \quad \delta E_p = E'_p - E_p \quad \text{and} \quad \delta E_k = E'_k - E_k.$$

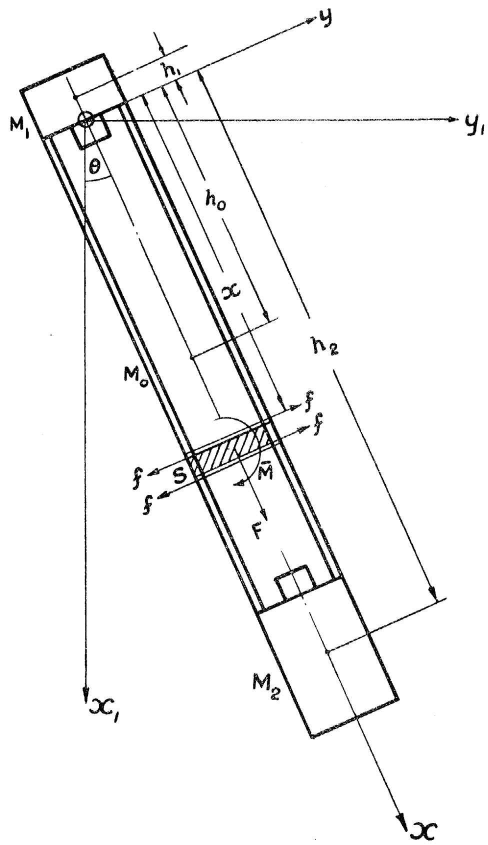

FIG. 12. Bending moment, etc. acting on a rigid pendulum.

If E'_{e_0}, E'_{p_0} and E'_{k_0} be the initial values (when the pendulum is vertically at rest) and if E'_{e_θ}, E'_{p_θ} and E'_{k_θ} be the values corresponding with a small displacement θ, then

$$E'_{e_\theta} + E'_{p_\theta} + E'_{k_\theta} = E'_{e_0} + E'_{p_0} + E'_{k_0}$$

or

$$\Big[E'_{e_\theta} + E'_{p_\theta} + E'_{k_\theta}\Big]_0^\theta = 0.$$

Now E_e (rigid pendulum) $= 0$; hence $\delta E_e = E'_e$ and

$$\Big[\delta E_e + (E_p + \delta E_p) + (E_k + \delta E_k)\Big]_0^\theta = 0.$$

But $\Big[E_p + E_k\Big]_0^\theta = 0$, being the equation of motion of a rigid pendulum; hence

$$\Big[\delta E_e + \delta E_p + \delta E_k\Big]_0^\theta = 0. \tag{1}$$

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Also, if ω is the angular velocity of the rigid pendulum and ω' that of the elastic pendulum, then, since δE_k may be written in the form

$$\tfrac{1}{2}\Sigma mr^2\delta(\omega^2)+\tfrac{1}{2}\omega^2\delta(\Sigma mr^2),$$

we have
$$\delta E_e+\delta E_p+\frac{\omega^2}{2}\delta(\Sigma mr^2) = -\tfrac{1}{2}\Sigma mr^2(\omega'^2-\omega^2),$$

or
$$\omega'^2-\omega^2 = -\frac{\delta E_e+\delta E_p+\dfrac{\omega^2}{2}\delta(\Sigma mr^2)}{\tfrac{1}{2}\Sigma mr^2}, \tag{2}$$

in which $\Sigma mr^2 = L\Sigma mh$, where L is the length of the equivalent simple pendulum, i.e.

$$L \equiv \frac{M_0 r_0^2+M_1 r_1^2+M_2 r_2^2}{M_0 h_0+M_1 h_1+M_2 h_2}.$$

From equation (2) the ratio of the periods of the elastic and rigid pendulums follows immediately. In order to calculate the terms δE_e, δE_p and δE_k it is necessary to know the form of the elastic pendulum at any instant, and this is completely determined if the bending moment on the pendulum rod and the force acting along its axis are known. These are assumed to be very closely the same as the bending moment and force, respectively, acting on a rigid pendulum of the same form as the (unstrained) elastic pendulum.

Suppose then, the rigid pendulum is displaced through the angle θ as in fig. 12, and let M be the bending moment at any section S of the rod distant x from O. Consider the rotation of the upper portion of the pendulum above the section S.

Then $Id^2\theta/dt^2$ is the moment of the forces about O.

Now the moment of inertia of the portion of the rod considered, about the point O, is given by

$$I = M_1 r_1^2+M_0\frac{x}{l}\left(\frac{x^2}{3}+C\right),$$

where C is the square of the radius of gyration of a cross-section of the rod about its neutral axis, which is mutually perpendicular to Ox and Oy:

$$C = \frac{1}{12}\frac{BD^3-bd^3}{BD-bd},$$

where B, b, etc., are the dimensions of the cross-section of the pendulum rod, in the usual notation for an I-section. But

$$\frac{d^2\theta}{dt^2} = \frac{g}{L}\sin\theta.$$

Hence
$$\left[M_1 r_1^2+M_0\frac{x}{l}\left\{\frac{x^2}{3}+C\right\}\right]\frac{g}{L}\sin\theta = I\frac{d^2\theta}{dt^2}.$$

But the clockwise moment of the forces about O may also be written

$$\left(M_1 h_1 + M_0 \frac{x^2}{2l}\right) g \sin\theta + M - xf,$$

where f is the shearing force at the section S.

Hence $$\left[M_1 r_1^2 + M_0 \frac{x}{l}\left\{\frac{x^2}{3} + C\right\}\right] \frac{g}{L} \sin\theta = \left(M_1 h_1 + M_0 \frac{x^2}{2l}\right) g \sin\theta + M - xf. \quad (3)$$

To determine f, consider the lower portion of the pendulum below S. The mass acceleration of the lower portion is

$$\left(M_2 h_2 + \frac{l^2 - x^2}{2l} M_0\right) \frac{d^2\theta}{dt^2}.$$

Hence $$\left(M_2 h_2 + \frac{l^2 - x^2}{2l} M_0\right) \frac{g}{L} \sin\theta = f + \left(M_0 \frac{l - x}{l} + M_2\right) g \sin\theta. \quad (4)$$

Then, solving (3) and (4) for M, we obtain

$$M = -M_0 lg \sin\theta \left[\frac{l}{6L}\left(\frac{x}{l}\right)^3 - \frac{1}{2}\left(\frac{x}{l}\right)^2 + A\left(\frac{x}{l}\right) + B\right], \quad (5)$$

where
$$\left.\begin{aligned} A &\equiv 1 - \frac{l}{2L} + \frac{M_2}{M_0}\left(1 - \frac{h_2}{L}\right) - \frac{C}{lL}, \\ B &\equiv \frac{M_1}{M_0}\left(\frac{h_1}{L} - \frac{r_1^2}{lL}\right), \\ \text{and} \quad C &\equiv \frac{1}{12} \frac{BD^3 - bd^3}{BD - bd}, \end{aligned}\right\} \quad (6)$$

giving the clockwise bending moment M at any distance x from O at the instant defined by the displacement θ.

To determine the axial tension F at any point of the rod distant x from O, we have that the total force on the section at S, in the direction Ox, due to the portion of the pendulum below S, is

$$M_0 g \cos\theta \left(\frac{l - x}{l} + \frac{M_2}{M_0}\right)$$

due to gravity, and $$M_0 g (\cos\theta - 1)\left(\frac{l^2 - x^2}{lL} + \frac{2h_2}{L}\frac{M_2}{M_0}\right)$$

due to centrifugal force, since $$\omega^2 = \frac{2g}{L}(\cos\theta - 1).$$

Hence $$F = M_0 g \cos\theta \left(\frac{l - x}{l} + \frac{M_2}{M_0}\right) + M_0 g (\cos\theta - 1)\left(\frac{l^2 - x^2}{Ll} + \frac{2h_2}{L}\frac{M_2}{M_0}\right). \quad (7)$$

(a) *Calculation of elastic strain energy of pendulum*

The strain energy e_1 of the pendulum rod due to bending only is given approximately by

$$e_1 = \frac{1}{2EI}\int_0^l (M)^2\,dx,$$

where I is the moment of inertia of the cross-section of the rod about its neutral axis, E is Young's modulus and M is the bending moment at a point distant x from the origin.

Now $$I = C(BD-bd),$$

or, if S is written throughout for the area of cross-section of the pendulum rod,

$$I = SC.$$

Hence $$e_1 = \frac{M_0^2 l^2 g^2 \sin^2\theta}{2ESC}\int_0^l f_1(x)\,dx,$$

where $f_1(x)$ is written for $\left[\frac{l}{6L}\left(\frac{x}{l}\right)^3 - \frac{1}{2}\left(\frac{x}{l}\right)^2 + A\left(\frac{x}{l}\right) + B\right]$ in equation (5).

For small amplitudes, $\sin\theta = \theta$, and the strain energy e_1, due to bending only, reduces to

$$e_1 = \frac{M_0^2 l^4 g^2 \theta^2}{2ESC}\int_0^l f_1^2(x)\,dx$$

$$= \frac{M_0^2 l^3 g^2 \theta^2}{2ESC}\left[\frac{1}{252}\left(\frac{l}{L}\right)^2 - \frac{1}{36}\left(\frac{l}{L}\right) + \frac{A}{15}\left(\frac{l}{L}\right) + \frac{B}{12}\left(\frac{l}{L}\right) - \frac{A}{4} + AB + B^2 + \frac{A^2}{3} - \frac{B}{3} + \frac{1}{20}\right]. \quad (8)$$

The strain energy e_2 due to stretching of the rod is given approximately by

$$e_2 = \frac{1}{2ES}\int_0^l F^2\,dx,$$

where F is the axial tension given by equation (7). Writing

$$f_2 \text{ for } \frac{l-x}{l} + \frac{M_2}{M_0}$$

and $$f_3 \text{ for } \frac{l^2-x^2}{lL} + \frac{2h_2}{L}\frac{M_2}{M_0},$$

then $$F^2 = M_0^2 g^2 \cos^2\theta f_2^2 + 2M_0^2 g^2(\cos\theta - 1) f_2 f_3,$$

neglecting the term in f_3^2, which depends on $(\cos\theta-1)^2$, and is therefore negligible.

Hence $$F^2 = M_0^2 g^2 (1-\theta^2) f_2^2 + 2M_0^2 g^2\left(-\frac{\theta^2}{2}\right) f_2 f_3$$

approximately. Now F^2 when the pendulum is vertical, i.e. at $\theta = 0$, is $M_0^2 g^2 f_2^2$; hence the increase in elastic energy of the rod due to stretching, when the displacement is θ, is given by

$$e_2 = -\frac{M_0^2 g^2 \theta^2}{2ES}\left[\int_0^l f_2^2\,dx + \int_0^l f_2 f_3\,dx\right]$$

$$= -\frac{M_0^2 g^2 l \theta^2}{2ES}\left[\frac{1}{3}+\frac{M_2}{M_0}+\left(\frac{M_2}{M_0}\right)^2+\frac{l}{L}\left\{\frac{1}{4}-\frac{1}{3}\left(1+\frac{M_2}{M_0}\right)+\left(\frac{1}{2}+\frac{M_2}{M_0}\right)\left(1+\frac{2h_2}{l}\frac{M_2}{M_0}\right)\right\}\right]. \quad (9)$$

The total elastic energy δE_e of the pendulum rod when the displacement is θ is then given by the algebraic sum of the two expressions (8) and (9), and may be expressed in the form

$$\delta E_e = (A_1 - A_2)\frac{g^2\theta^2}{ES}, \quad (10)$$

where A_1 and A_2 are numerical coefficients depending only on the dimensions and mass of the pendulum.* Their values are

$$A_1 = \frac{M_0^2 l^3}{2C}\left[\frac{1}{252}\frac{l^2}{L^2}-\frac{1}{36}\frac{l}{L}+\frac{A}{15}\frac{l}{L}+\frac{B}{12}\frac{l}{L}-\frac{A}{4}+AB+B^2+\frac{A^2}{3}-\frac{B}{3}+\frac{1}{20}\right]$$

and
$$A_2 = \frac{M_0^2 l}{2}\left[\frac{1}{3}+\frac{M_2}{M_0}+\left(\frac{M_2}{M_0}\right)^2+\frac{l}{L}\left\{\frac{1}{4}-\frac{1}{3}\left(1+\frac{M_2}{M_0}\right)+\left(\frac{1}{2}+\frac{M_2}{M_0}\right)\left(1+\frac{2h_2}{l}\frac{M_2}{M_0}\right)\right\}\right]. \quad (11)$$

In the case of the reversible pendulum, where $L = l$, these reduce to

$$A_1 \equiv \frac{M_0^2 l^3}{2C}\left[\frac{11}{420}-\frac{11}{60}A-\frac{B}{4}+AB+\frac{A^2}{3}+B^2\right]$$

and
$$A_2 \equiv \frac{M_0^2 l}{2}\left[\frac{3}{4}+\left(\frac{5}{3}+\frac{h_2}{l}\right)\frac{M_2}{M_0}+\left(1+\frac{2h_2}{l}\right)\left(\frac{M_2}{M_0}\right)^2\right]. \quad (11\,a)$$

(*b*) *Calculation of potential energy of pendulum*

The change in potential energy of the pendulum due to its elasticity may be expressed as the sum of two terms, thus

$$\delta E_p = \Sigma mg(\cos\theta - 1)\,\delta h + \Sigma mgh\,\delta(\cos\theta - 1),$$

in which the first term is determined by the stretching of the pendulum due to the force F, and the second term is derived from the lateral displacement of the pendulum due to the bending moment M.

From equation (5) for the bending moment at any point on the axis of the pendulum, we have, since $l = L$ for a reversible pendulum,

$$M = -M_0 lg\theta\left[\frac{1}{6}\left(\frac{x}{l}\right)^3-\frac{1}{2}\left(\frac{x}{l}\right)^2+A\left(\frac{x}{l}\right)+B\right].$$

* The elastic energy of the bobs has been neglected, since their cross-section is so large, compared with that of the rod, that they may be assumed to be rigid.

Hence
$$EI\frac{d^2y}{dx^2} = M_0 lg\theta\left[\frac{1}{6}\left(\frac{x}{l}\right)^3 - \frac{1}{2}\left(\frac{x}{l}\right)^2 + A\left(\frac{x}{l}\right) + B\right], \tag{12}$$

where y is the displacement of the element dx in the direction Oy, due to bending. Since the bending moment is clockwise, the sign of M is now chosen so that the positive direction of y is above the axis Ox, as shown in fig. 13. I is the moment of inertia of the cross-section of the pendulum rod about a diameter; hence, as before, $I = SC$, and

$$ESC\frac{d^2y}{dx^2} = M_0 lg\theta\left[\frac{1}{6}\left(\frac{x}{l}\right)^3 - \frac{1}{2}\left(\frac{x}{l}\right)^2 + A\left(\frac{x}{l}\right) + B\right]. \tag{13}$$

By integration

$$\frac{dy}{dx} = \frac{M_0 l^2 g\theta}{ESC}\left[\frac{1}{24}\left(\frac{x}{l}\right)^4 - \frac{1}{6}\left(\frac{x}{l}\right)^3 + \frac{A}{2}\left(\frac{x}{l}\right)^2 + B\frac{x}{l} + \left(\frac{1}{30} - \frac{A}{6} - \frac{B}{2}\right)\right], \tag{14}$$

and
$$y = \frac{M_0 l^3 g\theta}{ESC}\left[\frac{1}{120}\left(\frac{x}{l}\right)^5 - \frac{1}{24}\left(\frac{x}{l}\right)^4 + \frac{A}{6}\left(\frac{x}{l}\right)^3 + \frac{B}{2}\left(\frac{x}{l}\right)^2 + \frac{x}{l}\left(\frac{1}{30} - \frac{A}{6} - \frac{B}{2}\right)\right], \tag{15}$$

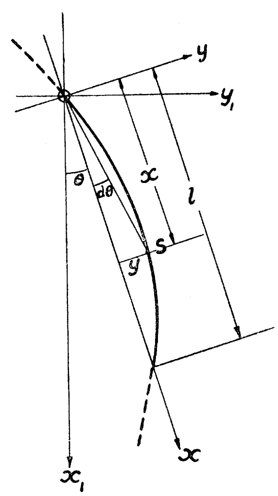

FIG. 13. Potential energy of elastic pendulum.

whence, since $d\theta = y/x$ (see fig. 13),

$$d\theta = \frac{M_0 l^2 g\theta}{ESC}\left[\frac{1}{120}\left(\frac{x}{l}\right)^4 - \frac{1}{24}\left(\frac{x}{l}\right)^3 + \frac{A}{6}\left(\frac{x}{l}\right)^2 + \frac{B}{2}\left(\frac{x}{l}\right) + \left(\frac{1}{30} - \frac{A}{6} - \frac{B}{2}\right)\right], \tag{16}$$

and $(\theta + d\theta)$ is the angle made at O by the element dx in its elastically displaced position.

The potential energy at the origin O being zero, then the potential energy of an element dx of the rigid pendulum rod at the instant defined by the displacement θ is $-S\rho gx\,dx\,(\cos\theta - 1) = +S\rho g\frac{1}{2}\theta^2 x\,dx$, where ρ is the density, and that for an element of the elastic pendulum rod is

$$\frac{S\rho gx}{2}dx(\theta + d\theta)^2 = \frac{S\rho gx\theta^2}{2}\left(1 + 2\frac{d\theta}{\theta}\right)dx.$$

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Hence the total potential energy of the elastic pendulum rod is

$$\frac{S\rho g\theta^2}{2}\int_0^l\left(x+2x\frac{d\theta}{\theta}\right)dx = M_0 g\frac{l}{2}\frac{\theta^2}{2}+S\rho g\theta^2\int_0^l x\frac{d\theta}{\theta}\,dx. \tag{17}$$

But the first term is the potential energy $-M_0 g\frac{1}{2}l(\cos\theta-1)$ of the rigid pendulum rod; hence the increase in potential energy of the rod due to bending only is given by

$$p_1 = S\rho g\theta^2\int_0^l x\frac{d\theta}{\theta}\,dx$$

$$= S\rho g\theta^2\int_0^l \frac{M_0 l^2 g}{ESC}\left[\frac{1}{120}\left(\frac{x}{l}\right)^5-\frac{1}{24}\left(\frac{x}{l}\right)^4+\frac{A}{6}\left(\frac{x}{l}\right)^3+\frac{B}{2}\left(\frac{x}{l}\right)^2+\frac{x}{l}\left(\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right)\right]dx$$

$$= \frac{M_0^2 l^3 g^2\theta^2}{ESC}\left[\frac{7}{720}-\frac{A}{24}-\frac{B}{12}\right], \tag{18}$$

which may be written in the form $$p_1 = B_1\frac{g^2\theta^2}{ES}, \tag{19}$$

where $$B_1 \equiv \frac{M_0^2 l^3}{C}\left[\frac{7}{720}-\frac{A}{24}-\frac{B}{12}\right]. \tag{20}$$

Now consider the upper bob M_1 in its elastically displaced position. The curvature of the bob near the point $x = 0$ due to the bending moment is appreciably less than the curvature of the rod near $x = 0$, since the cross-section of the bob has a greater moment of inertia than that of the rod. For the purpose of these calculations it is assumed that the bob is rigid, and that its axis is consequently straight, and continuous in direction with the axis of the pendulum rod near the point $x = 0$.

The slope of the axis of the bent pendulum rod at $x = 0$ is given by substituting $x = 0$ in equation (14), and we obtain

$$\left[\frac{dy}{dx}\right]_{x=0} = \frac{M_0 l^2 g\theta}{ESC}\left[\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right].$$

Hence the increase in potential energy of the upper bob due to bending of the pendulum rod is obtained by substituting dy/dx for $d\theta$ in the equation

$$p_2 = -M_1 g h_1\,\delta(\cos\theta-1) = M_1 g h_1\,\theta\,d\theta,$$

giving $$p_2 = \frac{M_0 M_1 g^2\theta^2 h_1 l^2}{ESC}\left[\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right], \tag{21}$$

which may be written in the form $$p_2 = B_2\frac{g^2\theta^2}{ES} \tag{22}$$

where $$B_2 \equiv \frac{M_0 M_1 h_1 l^2}{C}\left[\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right]. \tag{23}$$

Similarly, for the lower bob M_2,

$$\left[\frac{dy}{dx}\right]_{x=l} = \frac{M_0 l^2 g\theta}{ESC}\left[-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right],$$

whence $$p_3 = -M_2 g(h_2-l)\,\delta(\cos\theta-1) = M_2 g(h_2-l)\,\theta\,d\theta$$

or $$p_3 = \frac{M_0 M_2 g^2\theta^2(h_2-l)\,l^2}{ESC}\left[-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right],$$

which may be written $$p_3 = B_3\frac{g^2\theta^2}{ES}, \tag{24}$$

where $$B_3 \equiv \frac{M_0 M_2(h_2-l)\,l^2}{C}\left[-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right], \tag{25}$$

and the total change of potential energy due to the bending of the whole of the pendulum is

$$p_1+p_2+p_3 = (B_1+B_2+B_3)\frac{g^2\theta^2}{ES}.$$

Now consider the change of potential energy due to the stretching of the pendulum.

The force acting along the axis of the rigid pendulum in the direction Ox is given by equation (7) and may be written

$$F = M_0 g\cos\theta\,.f_2+M_0 g(\cos\theta-1)f_3. \tag{26}$$

The displacement of the element dx of the elastic pendulum in the direction Ox is Fdx/ES; hence the total displacement at the point S distant x from 0 is

$$\phi_2(x)\,l\cos\theta+\phi_3(x)\,l(\cos\theta-1), \tag{27}$$

where ϕ_2 is written for $$\frac{M_0 g}{lES}\int_0^x f_2\,dx$$

and ϕ_3 for $$\frac{M_0 g}{lES}\int_0^x f_3\,dx, \tag{28}$$

and the change in potential energy of the elastic rod, when the amplitude is θ, is given by

$$-M_0 gl\int_0^l [\phi_2\cos^2\theta+\phi_3\cos\theta(\cos\theta-1)]\,dx.$$

When the pendulum is vertical, $\theta = 0$, and the change in potential energy is

$$-M_0 gl\int_0^l \phi_2\,dx;$$

hence the change of potential energy p_4 due to elongation of the rod only is given by

$$p_4 = -M_0 gl(\cos\theta-1)\int_0^l [\phi_2(\cos\theta+1)+\phi_3\cos\theta]\,dx,$$

or, since θ is small
$$p_4 = \frac{M_0 lg\theta^2}{2}\int_0^l (2\phi_2+\phi_3)\,dx. \tag{29}$$

This reduces to
$$p_4 = \frac{M_0^2 lg^2\theta^2}{2ES}\left[\frac{13}{12}+\left(1+\frac{h_2}{l}\right)\frac{M_2}{M_0}\right], \tag{30}$$

which may be written
$$p_4 = B_4\frac{g^2\theta^2}{ES}, \tag{31}$$

where
$$B_4 \equiv M_0^2 l\left[\frac{13}{24}+\left(\frac{1}{2}+\frac{h_2}{2l}\right)\frac{M_2}{M_0}\right]. \tag{32}$$

The displacement at the lower end of the rod, i.e. at $x = l$, is, from equation (27),

$$\frac{M_0 gl}{ES}\left[\left(\frac{1}{2}+\frac{M_2}{M_0}\right)\cos\theta+\left(\frac{2}{3}+\frac{2h_2}{l}\frac{M_2}{M_0}\right)(\cos\theta-1)\right],$$

and the change in potential energy of the lower bob M_2 when the amplitude is θ is therefore

$$-M_2 gh_2\cos\theta\frac{M_0 g}{ES}\left[\left(\frac{1}{2}+\frac{M_2}{M_0}\right)\cos\theta+\left(\frac{2}{3}+\frac{2h_2}{l}\frac{M_2}{M_0}\right)(\cos\theta-1)\right].$$

When $\theta = 0$, the change in potential energy is

$$-M_2 gh_2\frac{M_0 g}{ES}\left(\frac{1}{2}+\frac{M_2}{M_0}\right).$$

Hence the change in potential energy p_5 of the lower bob M_2 is given by

$$p_5 = -M_2 gh_2(\cos\theta-1)\frac{M_0 g}{ES}\left[2\left(\frac{1}{2}+\frac{M_2}{M_0}\right)+\left(\frac{2}{3}+\frac{2h_2}{l}\frac{M_2}{M_0}\right)\right],$$

or
$$p_5 = \frac{M_0 M_2 h_2 g^2\theta^2}{ES}\left[\left(\frac{1}{2}+\frac{M_2}{M_0}\right)+\left(\frac{1}{3}+\frac{h_2}{l}\frac{M_2}{M_0}\right)\right],$$

which may be written
$$p_5 = B_5\frac{g^2\theta^2}{ES}, \tag{33}$$

where
$$B_5 \equiv M_0 M_2 h_2\left[\frac{5}{6}+\left(1+\frac{h_2}{l}\right)\frac{M_2}{M_0}\right]. \tag{34}$$

The total change of potential energy due to the elongation of the whole of the pendulum is then

$$p_4+p_5 = (B_4+B_5)\frac{g^2\theta^2}{ES},$$

and the total change of potential energy δE_p due to combined bending and stretching is given by

$$E_p = p_1+p_2+p_3+p_4+p_5,$$

which may also be expressed in the form

$$E_p = (B_1+B_2+B_3+B_4+B_5)\frac{g^2\theta^2}{ES}, \tag{35}$$

where the B terms are merely numerical coefficients.

(c) *Calculation of kinetic energy of pendulum*

A point P (see fig. 14) having co-ordinates $(x, 0)$ on the axis of the rigid pendulum becomes displaced to the point P' on the elastic pendulum.

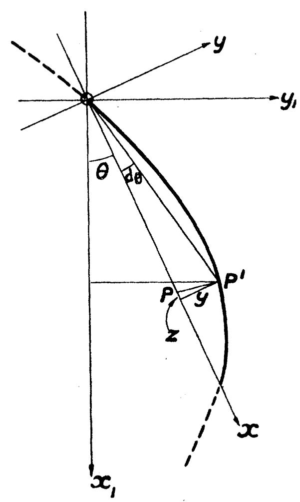

FIG. 14. Kinetic energy of elastic pendulum.

Let the co-ordinates of the point P', referred to the moving axes Ox, Oy, be $(x+z, y)$, the positive direction of y being in the same sense as before.

Now z can be determined from the force F, and y from the bending moment M.

The kinetic energy of an element dx at P' on the elastic pendulum is given by $\frac{\rho S\,dx}{2}v_1^2$, or the total kinetic energy of the rod by

$$\frac{\rho}{2}\int_0^V v_1^2\,dV,$$

where v_1 is the velocity of an element dV of the rod referred to the fixed vertical axis Ox_1.

Now $$v_1^2 = \dot{x}_1^2+\dot{y}_1^2,$$

but $$x_1 = (x+z)\cos\theta - y\sin\theta,$$

and $$y_1 = (x+z)\sin\theta + y\cos\theta,$$

whence $$v_1^2 = \left[x^2+2x\left(z+\frac{dy}{d\theta}\right)\right]\omega^2, \tag{36}$$

neglecting terms of the second and higher orders, and writing ω for $-d\theta/dt$. Thus the kinetic energy of the rod may be written

$$\frac{\rho S}{2}\omega^2\int_0^l x^2\,dx+\rho S\omega^2\int_0^l\left(xz+x\frac{dy}{d\theta}\right)dx.$$

But the first term is the kinetic energy of the rigid pendulum rod. Hence the change of kinetic energy of the elastic rod is

$$\rho S\omega^2\int_0^l xz\,dx+\rho S\omega^2\int_0^l\left(x\frac{dy}{d\theta}\right)dx. \tag{37}$$

In the first of these terms, the displacement z is given by equation (27) and it becomes $\rho S\omega^2\int_0^l xl\phi_2\,dx$ since θ is small.

But $$\phi_2=\frac{M_0 g}{lES}\int_0^x\left(\frac{l-x}{l}+\frac{M_2}{M_0}\right)dx=\frac{M_0 g}{ES}\left[\left(1+\frac{M_2}{M_0}\right)\frac{x}{l}-\frac{1}{2}\left(\frac{x}{l}\right)^2\right].$$

Hence $$\rho S\omega^2\int_0^l xz\,dx=\rho Sl\omega^2\left[\frac{1}{3}\left(1+\frac{M_2}{M_0}\right)-\frac{1}{8}\right]\frac{M_0 gl^2}{ES}$$

or $$k_1=\frac{M_0^2 gl^2\omega^2}{ES}\left[\frac{1}{3}\frac{M_2}{M_0}+\frac{5}{24}\right]. \tag{38}$$

In the second term of (37), $dy/d\theta$ is given by equation (15); thus $dy/d\theta=y/\theta$, and hence

$$\rho S\omega^2\int_0^l x\frac{dy}{d\theta}\,dx=\rho S\omega^2\int_0^l\frac{M_0^2 l^4 g}{ESC}\left[\frac{1}{120}\left(\frac{x}{l}\right)^6-\frac{1}{24}\left(\frac{x}{l}\right)^5+\frac{A}{6}\left(\frac{x}{l}\right)^4+\frac{B}{2}\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)^2\left(\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right)\right]dx$$

or $$k_2=\frac{M_0^2 l^4 g\omega^2}{ESC}\left[\frac{3}{560}-\frac{A}{45}-\frac{B}{24}\right]. \tag{39}$$

Consider now the kinetic energy of the upper bob M_1.* Its angular velocity anti-clockwise about the moving axes is $\frac{d}{dt}\left[\frac{dy}{dx}\right]_{x=0}$ or, from equation (14),

$$-\frac{M_0 l^2 g\omega}{ESC}\left[\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right],$$

since $\omega=-\dot{\theta}$. Hence its angular velocity about the fixed axis Ox_1 is

$$\omega_1=\omega\left[1+\frac{M_0 l^2 g}{ESC}\left(\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right)\right]$$

in the clockwise direction. Its kinetic energy is therefore

$$\tfrac{1}{2}M_1 r_1^2\omega^2+M_1 r_1^2\omega^2\left[\frac{M_0 l^2 g}{ESC}\left(\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right)\right].$$

* Assuming, as before, that its axis is straight.

The first term is the kinetic energy of the rigid bob M_1; hence the change of kinetic energy in the elastic pendulum due to the upper bob is given by

$$k_3 = M_1 r_1^2 \omega^2 \frac{M_0 l^2 g}{ESC}\left(\frac{1}{30}-\frac{A}{6}-\frac{B}{2}\right). \tag{40}$$

Similarly the anti-clockwise angular velocity of the lower bob M_2 about the moving axes is

$$\frac{d}{dt}\left[\frac{dy}{dx}\right]_{x=l} \text{ or } -\frac{M_0 l^2 g\omega}{ESC}\left(-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right)$$

from equation (14), and its angular velocity about the fixed axis Ox_1 is given by

$$\omega_2 = \omega\left[1+\frac{M_0 l^2 g}{ESC}\left(-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right)\right] \tag{41}$$

in the clockwise direction. The kinetic energy of the lower bob M_2 is then given by

$$\tfrac{1}{2}M_2 v^2 + \tfrac{1}{2}I_2 \omega_2^2,$$

where I_2 is the moment of inertia of M_2 about its centre of gravity, i.e.

$$I_2 = M_2 r_2^2 - M_2 h_2^2,$$

and v is the linear velocity of M_2 perpendicular to the axis Ox.

Now
$$v = (h_2 - l)\, n\omega + h_2\omega$$

where $\omega_2 \equiv \omega(1+n)$ is written for equation (41) and

$$v^2 = \omega^2[h_2^2 + 2h_2(h_2-l)\, n],$$

neglecting terms in n^2 since n is small. Hence the kinetic energy of the lower bob M_2 is given by

$$\frac{M_2}{2}\omega^2(h_2^2+2h_2^2 n-2h_2 ln)+\frac{M_2}{2}\omega^2(r_2^2-h_2^2)\,(1+2n) = \frac{M_2}{2}\omega^2[r_2^2+2n(r_2^2-lh_2)].$$

But the first term within the square brackets is the kinetic energy of the rigid bob M_2; hence the change in kinetic energy in the elastic pendulum due to the lower bob is given by

$$k_4 = M_2(r_2^2 - lh_2)\,\omega^2\frac{M_0 l^2 g}{ESC}\left(-\frac{11}{120}+\frac{A}{3}+\frac{B}{2}\right). \tag{42}$$

Finally the change in kinetic energy of the lower bob M_2 due to elongation of the pendulum rod is determined by the increase in r_2, which is very closely equal to the increase in h_2, due to the tension F. By equation (27) the displacement at $x = l$ is, to a first approximation, $\frac{M_0 gl}{ES}\left(\frac{1}{2}+\frac{M_2}{M_0}\right)$; hence the kinetic energy of M_2 is approximately

$$\frac{M_2}{2}\omega^2\left[h_2+\frac{M_0 gl}{ES}\left(\frac{1}{2}+\frac{M_2}{M_0}\right)\right]^2+\frac{M_2}{2}\omega^2(r_2^2-h_2^2) = \tfrac{1}{2}M_2 r_2^2\omega^2+\frac{M_0 M_2 gl\omega^2}{ES}\left(\frac{1}{2}+\frac{M_2}{M_0}\right). \tag{43}$$

The first term is the kinetic energy of the bob of the rigid pendulum; hence k_5, the increase in kinetic energy of the lower bob M_2, due to elongation of the pendulum rod, is given by

$$k_5 = \frac{M_0 M_2 g l h_2 \omega^2}{ES}\left(\frac{1}{2}+\frac{M_2}{M_0}\right). \tag{44}$$

The total change of kinetic energy of the elastic pendulum, due to changes in the moments of inertia of its component parts, is then given by

$$k_1+k_2+k_3+k_4+k_5.$$

But the total change of kinetic energy in the elastic pendulum may be written

$$\delta E_k = \frac{\omega^2}{2}\delta(\Sigma mr^2)+\tfrac{1}{2}\Sigma mr^2\,\delta(\omega^2), \tag{45}$$

in which the first term $\frac{1}{2}\omega^2\delta(\Sigma mr^2)$ is $k_1+k_2+k_3+k_4+k_5$. These terms $k_1, \ldots, k_5$ are all of the form $\frac{Ng\omega^2 l}{ES}$, where N is a numerical constant depending only on the mass and dimensions of the pendulum, and since

$$\omega^2 = \frac{2g}{L}(\cos\theta-1) = -\frac{g\theta^2}{l}$$

in the case of the reversible pendulum, then we may write

$$\frac{\omega^2}{2}\delta(\Sigma mr^2) = -(C_1+C_2+C_3+C_4+C_5)\frac{g^2\theta^2}{ES}. \tag{46}$$

The second term in the equation (45) may be written

$$\tfrac{1}{2}\Sigma mr^2\,\delta(\omega^2) = \frac{L}{2}(M_0h_0+M_1h_1+M_2h_2)\,(\omega'^2-\omega^2),$$

where ω' is the angular velocity of the elastic pendulum. Hence finally

$$\delta E_k = -(C_1+C_2+C_3+C_4+C_5)\frac{g^2\theta^2}{ES}+\frac{L}{2}(M_0h_0+M_1h_1+M_2h_2)\,(\omega'^2-\omega^2). \tag{47}$$

(*d*) *Final formulae*

The equation (2) may now be rewritten in the form

$$\omega'^2-\omega^2 = -\frac{\alpha+\beta+\gamma}{\frac{1}{2}\Sigma mr^2}\frac{g^2\theta^2}{ES}, \tag{48}$$

where
$$\alpha \equiv A_1-A_2,$$
$$\beta \equiv B_1+B_2+B_3+B_4+B_5,$$
$$\gamma \equiv -(C_1+C_2+C_3+C_4+C_5),$$

and since
$$\omega^2 = \frac{2g}{L}(\cos\theta - 1) = -\frac{g\theta^2}{L}$$

approximately, then
$$\left(\frac{\omega'}{\omega}\right)^2 - 1 = \frac{\alpha+\beta+\gamma}{\Sigma mr^2}\frac{2Lg}{ES}, \tag{49}$$

or
$$\left(\frac{\omega'}{\omega}\right)^2 = 1 + \frac{2g}{ES}\frac{\alpha+\beta+\gamma}{\Sigma mh},$$

whence
$$\frac{L}{L'} = 1 + \frac{2g}{ES}\frac{\alpha+\beta+\gamma}{\Sigma mh},$$

or
$$L = L'(1+K), \tag{50}$$

where L' is the length of the elastic pendulum, and where

$$K \equiv \frac{2g}{ES}\frac{\alpha+\beta+\gamma}{\Sigma mh}. \tag{51}$$

If K_1 and K_2 are the values of K corresponding with the pendulum upright and pendulum inverted respectively, and if h_1 and h_2 are now the distances of the centre of gravity of the pendulum from the knife-edge in the upright and in the inverted positions, respectively, then, in the usual notation,

$$\frac{gT_1^2}{\pi^2} = \frac{h_1^2+k^2}{h_1} + K_1 L$$

and
$$\frac{gT_2^2}{\pi^2} = \frac{h_2^2+k^2}{h_2} + K_2 L,$$

whence
$$\frac{g}{\pi^2}(h_1T_1^2 - h_2T_2^2) = h_1^2 - h_2^2 + (h_1K_1 - h_2K)\,L$$

or
$$\frac{g}{\pi^2}\frac{h_1T_1^2 - h_2T_2^2}{h_1 - h_2} \equiv \frac{gT^2}{\pi^2} = (h_1+h_2) + \left(\frac{h_1K_1 - h_2K_2}{h_1 - h_2}\right)L.$$

Hence the effective correction to the length of the equivalent pendulum, due to the elasticity of the pendulum rod, is

$$\frac{h_1K_1 - h_2K_2}{h_1 - h_2},$$

and equation (50) may be rewritten

$$L = L'\left(1 + \frac{h_1K_1 - h_2K_2}{h_1 - h_2}\right). \tag{52}$$

The whole of the data required to calculate the expression $(\alpha+\beta+\gamma)/\Sigma mh$ in equation (51) is derived from the masses and dimensions of the component parts of the pendulum, and since the values of the corrections K_1 and K_2 are small, only approximate values of the masses and dimensions need be used.

The data actually used, in inch-pound units, and the various stages in the computation of the corrections K_1 and K_2 are given in Table XIX, using the notation defined in this appendix.

TABLE XIX. DATA FOR ELASTICITY CORRECTIONS

(In.-lb. units.)

	Pendulum upright	Pendulum inverted
M_1	3·24	12·40
M_0	7·24	7·24
M_2	12·40	3·24
h_1	−0·76	−3·06
h_0	19·685	19·685
h_2	42·43	40·13
r_1^2	2·198	13·946
r_2^2	1805	1612
A	+0·3656	+0·49006
B	−0·00916	−0·14853
C	+2·0126	+2·0126
α	−12995	−415
β	+19710	+735
γ	−12270	−3270
$\alpha+\beta+\gamma$	−5555	−2950
Σmh	666·2	234·6
E/g	$10{\cdot}5\times10^6$	$10{\cdot}5\times10^{-6}$
S	1·7042	1·7042
	$K_1=-0{\cdot}9_3\times10^{-6}$	$K_2=-1{\cdot}4\times10^{-6}$

From the values of K_1 and K_2, the effective correction to the length of the pendulum, due to its elasticity is

$$K=\frac{(-0{\cdot}93)\times(73{\cdot}94)-(-1{\cdot}4)\,(26{\cdot}05)}{47{\cdot}89}=-0{\cdot}7 \text{ part in a million.}$$

The importance of considering the effect of bending as well as the effect of elongation of the elastic pendulum is shown in the following table, in which the values calculated for the actual pendulum used are compared with the values calculated for an earlier design of reversible pendulum of the same length having a solid cylindrical rod of diameter $\frac{7}{8}$ in. and cylindrical bobs.

ELASTICITY CORRECTIONS (UNIT 10^{-6})

Form of pendulum	True elasticity corrections			Corrections due to elongation only		
	Upright K_1	Inverted K_2	Effective K	Upright	Inverted	Effective
(1) Cylindrical rod $\frac{7}{8}$ in. diam. with cylindrical bobs. Total weight 21 lb. approx.	−17	−53	**+13**	+1·5	+0·8	**+2·1**
(2) Pendulum with I-section rod and rectangular bobs. Total weight 22·9 lb. approx.	−0·9	−1·4	**−0·7**	+0·9	+0·4	**+1·2**

The simple calculation of the elongation of this rod due to its own weight and to the tension due to the lower bob indicated an elasticity correction of about two parts in a million, and it was not until the above calculations were carried out that it was realized that the true elasticity correction for such a pendulum was more than six times as large, the change of shape of the pendulum rod due to bending having a much larger effect on the period than the mere elongation due to the weight of the pendulum. In the case of the pendulum with the rod of I-section, the discrepancy is not so great; the actual correction is rather smaller than that due to elongation alone.

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